JOURNAL OF MULTIVARIATE ANALYSIS 51, 277–293 (1994)

Comparing Empirical Likelihood and Bootstrap Hypothesis Tests

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A comparison between empirical likelihood and bootstrap tests for a mean parameter against a series of local alternative hypotheses is made by developing Edgeworth expansions for the power functions of the two tests. For univariate and bivariate cases, practical rules are proposed for choosing the more powerful test.

1. INTRODUCTION

Empirical likelihood, introduced by Owen [13, 14], is a nonparametric method of inference with sampling properties similar to those of the bootstrap. However, instead of assigning equal probabilities n^{-1} to all data values, empirical likelihood places arbitrary probabilities on the data points, say p_i on the ith data value. The weights p_i are chosen by profiling a multinomial likelihood supported on the sample, and empirical likelihood confidence regions are constructed by contouring this multinomial likelihood. An attractive feature of empirical likelihood is that it produces confidence regions whose shapes and orientations are determined entirely by the data, and which have coverage accuracy at least comparable with those of bootstrap confidence regions. Its coverage properties have been examined by Hall and La Scala [10] and DiCiccio *et al.* [8] for the case of smooth function of a mean of independent and identically distributed random variables; by Chen and Hall [4] for the case of quantiles; by Owen [15] and Chen [5, 6] for the regression case, and by Kolaczyk [12] for the case of generalized linear models.

Let $X_1, ..., X_n$ be an independent and identically distributed (i.i.d.) random sample of p dimension from unknown distribution with mean

Received August 17, 1992; revised October 28, 1993.

AMS 1980 Subject Classification: 62E20.

Key words and phrases: Bootstrap, confidence region, Edgeworth expansion, empirical likelihood, hypothesis test, power of test.

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0047-259X/94 $6.00

parameter μ and covariance matrix Σ. We consider using empirical likelihood and bootstrap methods to test the null hypothesis $H_0: \mu = \mu_0$ against a series of local alternatives $H_n: \mu = \mu_0 + n^{-1/2}\Sigma^{1/2}\tau$, where both μ_0 and τ are constant p dimensional vectors. Empirical likelihood and bootstrap hypothesis tests for H_0 can be formulated from the well known duality between a confidence region and a hypothesis test.

Owen [14] showed in the i.i.d. sample mean case that the power of an α-level empirical log likelihood ratio test is asymptotically $P\{\chi_p^2(\|\tau\|^2) > \chi^2_{p,1-\alpha}\}$, where $\chi_p^2(\|\tau\|^2)$ is the noncentral chi-squared random variable with p degrees of freedom and noncentral term $\|\tau\|^2$, and $\chi^2_{p,1-\alpha}$ is the $1-\alpha$ upper percentile of the central chi-squared distribution χ^2_p. However, it is not difficult to show that the corresponding bootstrap test also achieves the same asymptotic power. Thus, to really compare the powers of the empirical likelihood and bootstrap tests, we have to develop higher order expansions for the powers of these two tests, which will give us some insight into the problem. Surprisingly, so far nobody has given a higher order expansion for the power function of bootstrap test neither.

The aim of this paper is to compare the powers of empirical likelihood and bootstrap tests by developing expansions for their powers. In Section 2 we give definitions for empirical likelihood and bootstrap tests. After developing higher order expansions for the power functions in Section 3, we propose in Section 4 two rules for practically choosing between empirical likelihood and bootstrap tests for univariate and bivariate cases. In the univariate case, the rule says that the empirical likelihood test is more powerful than the corresponding bootstrap test when $\tau\alpha_3 > 0$, and vice versa when $\tau\alpha_3 < 0$, where α_3 is the population skewness parameter. For higher dimensional cases, similar rules may be developed. In Section 5 we present simulation studies.

2. Empirical Likelihood and Bootstrap Hypothesis Tests

Let $X_1, ..., X_n$ be a p dimension i.i.d. sample from unknown distribution F with mean μ and covariance matrix Σ. We want to test null hypothesis $H_0: \mu = \mu_0$ against a series of local alternatives $H_n: \mu = \mu_0 + n^{-1/2}\Sigma^{1/2}\tau$, where μ_0 and τ are p dimension constant vectors.

Put $Z_i = \Sigma^{-1/2}(x_i - \mu)$ and Z_i^j as the jth component of Z_i. We define

$$\alpha^{j_1 j_2 \cdots j_k} = E(Z_i^{j_1} \cdots Z_i^{j_k}),$$

$$A^{j_1 j_2 \cdots j_k} = n^{-1} \sum Z_i^{j_1} \cdots Z_i^{j_k} - \alpha^{j_1 j_2 \cdots j_k},$$

as the standarized multivariate moments of $X_1, ..., X_n$. Note that $\alpha^j = 0$ and $\alpha^{jk} = \delta^{jk}$ where δ^{jk} is the Kronecker delta.

2.1. *Empirical Likelihood Tests*

Write $p_1, p_2, ..., p_n$ for nonnegative numbers adding to unity. Then, the empirical log likelihood ratio for μ is defined to be

$$l(\mu) = -2 \min_{\sum p_i x_i = \mu} \sum_{i=1}^{n} \log(np_i).$$

Based on the nonparametric version of Wilks' Theorem given by Owen [14], a $1-\alpha$ level confidence region for μ is defined as $I_{1-\alpha} = \{\mu \mid l(\mu) < c_\alpha\}$, where c_α is chosen from χ_p^2 tables such that $P(\chi_p^2 > c_\alpha) = \alpha$. According to the duality between confidence regions and hypothesis tests, we define an α-level empirical likelihood test for the null hypothesis H_0 to be

$$\phi_e = \begin{cases} 1, & \text{if } \; l(\mu_0) > c_\alpha; \\ 0, & \text{otherwise.} \end{cases}$$

By Wilks' Theorem the asymptotic significant level of ϕ_e is α. Let us define "the first type accuracy" as the difference between the actual and nominal significant levels of a test. Using the results given by Hall and La Scala [10], we may show that $P(\phi_e = 1 \mid H_0) = \alpha + O(n^{-1})$, which means the first type accuracy of the empirical likelihood test ϕ_e is of order n^{-1}. Since the empirical likelihood confidence regions are Bartlett correctable in this case, as shown by DiCiccio *et al.* [8], we may define Bartlett-corrected empirical likelihood test to be

$$\phi_{ec} = \begin{cases} 1, & \text{if } \; l(\mu_0) > c_\alpha(1 + \hat{\beta}/n); \\ 0, & \text{otherwise,} \end{cases}$$

where $\hat{\beta} = p^{-1}(\hat{\alpha}^{iijkk}/2 - \hat{\alpha}^{jkl}\hat{\alpha}^{jkl}/3)$ is the empirical Bartlett factor, and $\hat{\alpha}^{iijkk}$ and $\hat{\alpha}^{jkl}$ are the usual moment estimates of α^{iijkk} and α^{jkl} respectively. Let $\beta = p^{-1}(\alpha^{iijkk}/2 - \alpha^{jkl}\alpha^{jkl}/3)$ be the theoretical Bartlett factor. Clearly we have $\hat{\beta} = \beta + O_p(n^{-1/2})$. Since $P(\phi_{ec} = 1 \mid H_0) = \alpha + O(n^{-2})$, the first type accuracy of the corrected empirical likelihood test ϕ_{ec} is of order n^{-2}, which is of the same order with that of bootstrap test as will be shown shortly.

2.2 *Bootstrap Test*

Let define $\bar{x} = n^{-1} \sum x_i$ and $\hat{\Sigma} = n^{-1} \sum (x_i - \bar{x})(x_i - \bar{x})^T$ be the sample mean and sample covariance matrix respectively. To give an α-level bootstrap test of H_0, let $\bar{x}^*$ and $\hat{\Sigma}^*$ be the bootstrap version of $\bar{x}$ and $\hat{\Sigma}$ respectively, computed from a resample χ^* instead of the entire sample $\chi = \{X_1, ..., X_n\}$. Put $S(\tau) = n^{1/2}\hat{\Sigma}^{-1/2}(\bar{x} - \mu + n^{-1/2}\Sigma^{1/2}\tau)$. We define a bootstrap test of H_0 to be

$$\phi_b = \begin{cases} 1, & \text{if } \; S^T(\tau)\, S(\tau) > c_\alpha^*; \\ 0, & \text{otherwise,} \end{cases}$$

where c_α^* is determined by $P\{n(\bar{x}^*-\bar{x})^T \hat{\Sigma}^{*-1}(\bar{x}^*-\bar{x})>c_\alpha^* \mid \chi\}=\alpha$, and c_α^* can be determined by Monte Carlo simulations. It has been pointed out by Hall [9] that $P(\phi_b=1 \mid H_0)=\alpha+O(n^{-2})$, which means the first type accuracy of the bootstrap test ϕ_b is of order n^{-2}.

Owen [14] showed that for our current null and alternative hypothesis setting, the power of the uncorrected empirical likelihood test ϕ_e (also the corrected empirical likelihood test ϕ_{ec} as shown in Section 3) converges to $P\{\chi_p^2(\|\tau\|^2)>\chi_{p,1-\alpha}^2\}$, where $\chi_p^2(\|\tau\|^2)$ is the noncentral chi-squared random variable with noncentral term $\|\tau\|^2$. It is not difficult to show that the bootstrap test achieves the same asymptotic power as well. In order to compare the power performances of these tests we have to find higher order expansions for the power functions of the empirical likelihood and bootstrap tests. To make the comparison fairly, we should only compare the corrected empirical likelihood test ϕ_{ec} with the bootstrap test ϕ_b, since both have the same first type accuracy of order n^{-2}. In theory we could adjust the test's level so that they are exactly equal. However, from a practical point of view a difference of order n^{-2} between the levels of the tests is fair enough to make our comparison. In the rest of this paper, when we say the empirical likelihood test we mean the Bartlett-corrected test ϕ_{ec}.

Before we finish this section, we should mention that the shape of the rejection region of the empirical likelihood test is determined automatically by the sample itself, whereas that of the bootstrap test is subjectively given by us as the complement of an elliptical region. This is an advantage of empirical likelihood over the bootstrap.

3. Power Expansions

In this section we calculate the powers of the empirical likelihood and bootstrap tests of null hypothesis $H_0: \mu=\mu_0$ against $H_n: \mu=\mu_0+n^{-1/2}\Sigma^{1/2}\tau$. Since analytic expressions for these power functions are difficult to attain, we have to develop expansions for them.

Let $\text{pow}(\phi_{ec}; \tau)$ and $\text{pow}(\phi_b, \tau)$ denote the powers of the α-level empirical likelihood tests ϕ_{ec} and the bootstrap test ϕ_b respectively, under the alternative hypothesis H_n. We shall calculate them one by one.

3.1. *Power of* ϕ_{ec}

According to the definition of power of a test, we have

$$\begin{aligned}\text{pow}(\phi_{ec}; \tau) &= P(\phi_{ec}=1 \mid H_n)\\ &= P\{l(\mu_0)>\tilde{c}_\alpha \mid \mu=\mu_0+n^{-1/2}\Sigma^{1/2}\tau\}\\ &= P\{l(\mu-n^{-1/2}\Sigma^{1/2}\tau)>\tilde{c}_\alpha\},\end{aligned}$$

where $\tilde{c}_\alpha = c_\alpha(1+\hat{\beta}/n)$. To calculate $\text{pow}(\phi_{ec};\tau)$ we first set up a Taylor expansion for $l(\mu - n^{-1/2}\Sigma^{1/2}\tau)$, from which an Edgeworth expansion of $\text{pow}(\phi_{ec};\tau)$ will be derived. By the definition of empirical likelihood,

$$l(\mu - n^{-1/2}\Sigma^{1/2}\tau) = -2 \min_{\sum p_i x_i = \mu - n^{1/2}\Sigma^{1/2}\tau} \sum \log(np_i)$$
$$= -2 \min_{\sum p_i Z_i = -n^{-1/2}\tau} \sum \log(np_i),$$

where $Z_i = \sum^{1/2}(x_i - \mu)$. Using (3.7) of DiCiccio *et al.* [7], we have

$$\begin{aligned} n^{-1} l(\mu - n^{-1/2}\Sigma^{1/2}\tau) = {} & (A + n^{-1/2}\tau)^j (A + n^{-1/2}\tau)^j \\ & - \{A^{jk} + n^{-1/2}\tau^j A^k\,[2] + n^{-1}\tau^j\tau^k\} \\ & \times (A + n^{-1/2}\tau)^j (A + n^{-1/2}\tau)^k \\ & + \tfrac{2}{3}(\alpha^{jkl} + A^{jkl} + n^{-1/2}\tau^j\delta^{kl}\,[3] - 2\alpha^{jkm}A^{lm}) \\ & \times (A + n^{-1/2}\tau)^j (A + n^{-1/2}\tau)^k (A + n^{-1/2}\tau)^l \\ & + (\alpha^{jkn}\alpha^{lmn} - \tfrac{1}{2}\alpha^{jklm})(A + n^{-1/2}\tau)^j (A + n^{-1/2}\tau)^k \\ & \times (A + n^{-1/2}\tau)^l (A + n^{-1/2}\tau)^m \\ & + A^{jl}A^{kl}(A + n^{-1/2}\tau)^j (A + n^{-1/2}\tau)^k + O_p(n^{-5/2}), \end{aligned} \tag{3.1}$$

with $\tau^j A^k[2] = \tau^j A^k + \tau^k A^j$ and the same rule applies for $\tau^j\delta^{kl}[3]$. Here we use the summation convention according to which if an index occurs more than once in an expression then summation over that index is understood. From (3.1) we can derive the following signed root decomposition for $l(\mu - n^{-1/2}\Sigma^{1/2}\tau)$:

$$l(\mu - n^{-1/2}\Sigma^{1/2}\tau) = n(R_1 + R_2 + R_3)^T (R_1 + R_2 + R_3) + O_p(n^{-5/2}),$$

where $R_l = O_p(n^{-l/2})$ for $l = 1, 2, 3$, and

$$\begin{aligned} R_1^j = {} & (A + n^{-1/2}\tau)^j, \\ R_2^j = {} & -\tfrac{1}{2}A^{jk}((A + n^{-1/2}\tau)^k + \tfrac{1}{3}\alpha^{jkl}(A + n^{-1/2}\tau)^k (A + n^{-1/2}\tau)^l, \\ R_3^j = {} & (-\tfrac{1}{2}\tau^j\tau^k n^{-1} + \tfrac{3}{8}A^{jl}A^{kl} - \tfrac{1}{2}\tau^j A^k[2]n^{-1/2})(A + n^{-1/2}\tau)^k \\ & + \{\tfrac{1}{3}(A^{jkl} + n^{-1/2}\tau^j\delta^{kl}[3]) - \tfrac{5}{6}\alpha^{jkm}A^{lm}\}(A + n^{-1/2}\tau)^j (A + n^{-1/2}\tau)^k \\ & + (\tfrac{4}{9}\alpha^{jkn}\alpha^{lmn} - \tfrac{1}{4}\alpha^{jklm})(A + n^{-1/2}\tau)^k (A + n^{-1/2}\tau)^l (A + n^{-1/2}\tau)^m. \end{aligned} \tag{3.2}$$

Put $R=R_1+R_2+R_3$. Let $k_l^{j_1,\ldots,j_l}$ denote the joint lth order cumulant of $n^{1/2}R$. Calculations show that

$$\begin{aligned} k_1^j &= \tau^j + k_{11}^j n^{-1/2} + k_{12}^j n^{-1} + O(n^{-3/2}),\\ k_2^{jk} &= \delta^{jk} + k_{21}^{jk} n^{-1/2} + k_{22}^{jk} n^{-1} + O(n^{-3/2}),\\ k_3^{jkl} &= k_{32}^{jkl} n^{-1} + O(n^{-3/2}),\ k_4^{jklm} = O(n^{-3/2}),\\ k_l^{j_1,\ldots,j_l} &= 0(n^{-3/2}), \qquad \text{for} \quad l \geqslant 5, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} k_{11}^j &= \left(\frac{1}{3}\alpha^{jlm}\tau^l\tau^m - \frac{1}{6}\alpha^{jkk}\right), \qquad k_{21}^{jk} = \frac{1}{3}\alpha^{jkl}\tau^l,\\ k_{12}^j &= \frac{1}{2}\tau^j\tau^k\tau^k + \frac{7}{24}\tau^k\alpha^{jkll} + \left(\frac{4}{9}\alpha^{jkn}\alpha^{lmn} - \frac{1}{4}\alpha^{jklm}\right)\tau^k\tau^l\tau^m\\ &\quad + \left(-\frac{p}{6} - \frac{5}{24}\right)\tau^j - \frac{7}{18}\alpha^{jkm}\alpha^{mll}\tau^k + \frac{1}{18}\alpha^{jkm}\alpha^{klm}\tau^l,\\ &\quad + \left(\frac{8}{9}\alpha^{jkm}\alpha^{lmn} - \frac{1}{9}\alpha^{jmn}\alpha^{klm} - \frac{7}{12}\alpha^{jkln}\right)\tau^l\tau^n,\\ k_{32}^{jkl} &= -\frac{1}{2}\alpha^{jklm}\tau^m + \frac{5}{36}\tau^n\alpha^{jmn}\alpha^{klm}[3]. \end{aligned} \tag{3.4}$$

Note that the last formula in (3.3) is obtained from the general results given by James and Mayne [11].

Let ϕ be the density of $N(0, I_p)$, $H_l(v_{j_1}, \ldots, v_{j_l})$ be the lth order multivariate Chebyshev–Hermite polynomials defined by Barndorff-Nielson and Cox [1] and $\mathscr{D}_\tau(x) = \{v \mid \|v+\tau\| > x\}$. Then, we define

$$\begin{aligned} E_2(x,\tau) &= \int_{\mathscr{D}_\tau(x)} \{k_{11}^j v_j + \tfrac{1}{2}k_{21}^{jk}(v_j v_k - \delta^{jk})\}\,\phi(v)\,dv,\\ E_3(x,\tau) &= \int_{\mathscr{D}_\tau(x)} \{k_{12}^j v_j + \tfrac{1}{2}k_{22}^{jk} + k_{11}^j k_{11}^k)(v_j v_k - \delta^{jk})\}\,\phi(v)\,dv,\\ &\quad + \int_{\mathscr{D}_\tau(x)} (\tfrac{1}{6}k_{32}^{jkl} + \tfrac{1}{2}k_{11}^j k_{21}^{kl})\,H_3(v_j, v_k, v_l)\,\phi(v)\,dv,\\ &\quad + \int_{\mathscr{D}_\tau(x)} \tfrac{1}{8}k_{21}^{jk}k_{21}^{lm} H_4(v_j, v_k, v_l, v_m)\,\phi(v)\,dv. \end{aligned} \tag{3.5}$$

Put $\bar{U} = (A^1, \ldots, A^p, A^{11}, \ldots, A^{pp}, A^{111}, \ldots, A^{ppp})^T = n^{-1}\sum U_i$ where

$$U_i = (Z_p^1, \ldots, Z_p^p, Z_p^1 Z_p^1, \ldots, Z_p^p Z_p^p, Z_p^1 Z_p^1, \ldots, Z_1^p Z_i^p Z_i^p).$$

Note that only A^{jk} and A^{jkl} with $j \leqslant k \leqslant l$ appear in $\bar{U}$. With above preparations we are able to prove the following which lead to an Edgeworth expansion for $\text{pow}(\phi_{ec}, \tau)$.

THEOREM 3.1. *Assume that U_1 has finite fifth moments and the characteristic function g_1 of U_1 satisfies the Cramér's condition, that is, $\sup_{\|t\| > b} |g_1(t)| < 1$. Then, for any $x > 0$,*

$$P\{l(\mu - n^{-1/2}\Sigma^{1/2}\tau) > x\} = P\{\chi_p^2(\|\tau\|) > x\} + E_2(x, \tau)\, n^{-1/2} + E_3(x, \tau)\, n^{-1} + O(n^{-3/2}).$$

Proof. Define

$$\begin{aligned}\Pi(v) = 1 &+ n^{-1/2}\{k_{11}^{j} v_j + \tfrac{1}{2} k_{21}^{jk}(v_j v_k - \delta^{jk})\} + n^{-1}\{k_{12}^{j} v_j \\ &+ \tfrac{1}{2}(k_{22}^{jk} + k_{11}^{j} k_{11}^{k})(v_j v_k - \delta^{jk}) + (\tfrac{1}{6} k_{32}^{jkl} + \tfrac{1}{2} k_{11}^{j} k_{21}^{kl})\, H_3(v_j, v_k, v_l) \\ &+ \tfrac{1}{6} k_{21}^{jk} k_{21}^{lm} H_4(v_j, v_k, v_l, v_m)\},\end{aligned}$$

where the k's are given by (3.4). From (3.3) and (3.4), a formal Edgeworth expansion for the distribution function of $n^{1/2}R$ can be constructed as follows,

$$P(n^{1/2}R < x) = \int_{-\infty}^{x} \Pi(v)\, \phi(v)\, dv + O(n^{-3/2}). \tag{3.6}$$

Accepting that expansion (3.6) may be justified, we establish an Edgeworth expansion for the distribution of $l(\mu - n^{-1/2}\Sigma^{1/2}\tau)$ as follows,

$$\begin{aligned}P\{l(\mu - n^{-1/2}\Sigma^{1/2}\tau) > x\} &= P(nR^T R > x) + O(n^{-3/2}) \\ &= \int_{\mathscr{D}_\tau(x)} \Pi(v)\, \phi(v)\, dv + O(n^{-3/2}) \\ &= P\{\chi_p^2(\|\tau\|) > x\} + E_2(x, \tau)\, n^{-1/2} \\ &\quad + E_3(x, \tau)\, n^{-1} + O(n^{-3/2}),\end{aligned}$$

where $E_2(x, \tau)$ and $E_3(x, \tau)$ are given in (3.5).

It remains to check that expansion (3.6) is valid. Since

$$\bar{U} = (A^1, ..., A^p, A^{11}, ..., A^{pp}, A^{111}, ..., A^{ppp})^T.$$

We see that $\bar{U}$ is the mean of i.i.d. random vectors with mean 0, and there exists a smooth functions h such that $R = h(\bar{U})$. Thus, we can justify the expansion (3.6) by using the result given by Bhattacharya and Ghosh [2]. Therefore the theorem is valid. ∎

From Theorem 3.1 and using the delta method, we have

$$\begin{aligned}\text{pow}(\phi_{ec};\tau) &= P\{l(\mu - n^{-1/2}\Sigma^{1/2}\tau) > c_\alpha(1+\hat{\beta}/n)\}\\ &= P\{l(\mu - n^{-1/2}\Sigma^{1/2}\tau) > c_\alpha(1+\beta/n)\} + O(n^{-3/2})\\ &= P\{\chi_p^2(\|\tau\|) > c_\alpha\} + E_2(c_\alpha,\tau)\,n^{-1/2}\\ &\quad + \{E_3(c_\alpha,\tau) - \beta g_{p\tau}(c_\alpha)\}\,n^{-1} + O(n^{-3/2}),\end{aligned} \tag{3.7}$$

where $g_{p\tau}$ is the density of $\chi_p^2(\|\tau\|)$ distribution. Substituting k_{11}^j and k_{21}^j in (3.4) into the expression for E_2, we obtain

$$E_2(c_\alpha,\tau) = \int_{\mathscr{S}_r(c_\alpha)} (\tfrac{1}{3}\alpha^{jlm}\tau^l\tau^m - \tfrac{1}{6}\alpha^{jkk})\,v_j + \tfrac{1}{6}\alpha^{jkl}\tau^l(v_jv_k - \delta^{jk})\}\,\phi(v)\,dv. \tag{3.8}$$

Thus the second order term of the power of the empirical likelihood test depends on population skewness parameter α^{jlm}, on τ and on sample size n.

Remark. If $X_1, ..., X_n$ are independent but not identically distributed with same mean parameter μ, we can modify the definitions of Σ and $\alpha^{j_1j_2\cdots j_k}$ by $\Sigma = n^{-1}\Sigma\,\text{Cov}(x_i)$ and $\alpha^{j_1j_2\cdots j_k} = n^{-1}\Sigma\, E(Z_i^{j_1}\cdots Z_i^{j_k})$, and define $A^{j_1,\ldots,j_k}$ and $U_1, ..., U_n$ in the same way of the i.i.d. case. To obtain the expansion for the power function in Theorem 3.1, we have to replace the conditions of the theorem with the following one:

(i) The smallest eigenvalue of $n^{-1}\sum \text{Cov}(U_i)$ is bounded away from zero

(ii) $\sup_n n^{-1}\sum_{j=1}^n E\,\|U_j\|^5 < \infty$.

(iii) $\lim_{n\to\infty} n^{-1}\sum_{j=1}^n \int_{\|U_j\| > \tau n^{1/2}} \|U_j\|^5 = 0$, for every positive τ.

(iv) The characteristic function g_j of U_j satisfies the Cramér's condition, $\lim\sup_{j\to\infty}\sup_{\|t\|>b} |g_j(t)| < 1$, for every positive b.

Then, we may get Theorem 3.1 and (3.7) for this non-i.i.d. case by using Theorem 20.6 of Bhattacharya and Rao [3] and Skovgaard [16]. For details see the proof of Theorem 2.1 of Chen [5].

3.2. *Power of ϕ_b*

In this subsection we give an expansion for the power of the bootstrap test ϕ_b as we have done for the empirical likelihood test in subsection 3.1. According to definition,

$$\begin{aligned}\text{pow}(\phi_b;\tau) &= P(\phi_b = 1 \mid H_n)\\ &= P\{S^T(\tau)\,S(\tau) > c_\alpha^*\},\end{aligned} \tag{3.9}$$

where c_α^* is determined by equation $P\{n(\bar{x}^*-\bar{x})^T\hat{\Sigma}^{*-1}(\bar{x}^*-\bar{x})>c_\alpha^*\mid\chi\}=\alpha$. Let $\xi_l^{j_1,\ldots,j_l}$ denote the joint lth order cumulant of $S(\tau)$. Calculations reveal that

$$\begin{aligned}
&\xi_1^j=\tau^j+\xi_{11}^j n^{-1/2}+\xi_{12}^j n^{-1}+O(n^{-1/2}),\\
&\xi_2^{jk}=\delta^{jk}+\xi_{21}^{jk}n^{-1/2}+\xi_{22}^{jk}n^{-1}+O(n^{-3/2}),\\
&\xi_3^{jkl}=\xi_{31}^{jkl}n^{-1/2}+\xi_{32}^{jkl}n^{-1}+O(n^{-3/2}),\\
&\xi_4^{jklm}=\xi_{41}^{jklm}n^{-1}+O(n^{-3/2}),\\
&\xi_l^{j_1,\ldots,j_l}=O(n^{-3/2}),\qquad\text{for}\quad l\geqslant 5.
\end{aligned}\tag{3.10}$$

where

$$\begin{aligned}
&\xi_{11}^j=-\tfrac{1}{2}\alpha^{jkk},\ \xi_{12}^j=\tfrac{1}{2}\tau^j+\tfrac{3}{8}\tau^l(\alpha^{jkkl}-\delta^{jl}),\\
&\xi_{21}=-\alpha^{jkm}\tau^m,\\
&\xi_{22}^{jk}=(p+2)\,\delta^{jk}+\alpha^{jlm}\alpha^{klm}+\tfrac{3}{4}\alpha^{jkm}\alpha^{llm}+\tfrac{1}{4}(\alpha^{jklm}-\delta^{jk})\,\tau^l,\\
&\xi_{31}^{jkl}=-2\alpha^{jkl},\ \xi_{32}^{jkl}=\tau^j\delta^{kl}[3]+\tfrac{5}{4}\tau^n\alpha^{jkm}\alpha^{lmn}[3],\\
&\xi_{41}^{jklm}=-2\alpha^{jklm}+4\alpha^{jmn}\alpha^{kln}[3]+4\delta^{jm}\delta^{kl}.
\end{aligned}\tag{3.11}$$

Let g_p be the density function of the χ_p^2 distributions, $\mathscr{D}=\{v\mid\|v\|\geqslant c_\alpha\}$ where $v=(v_1,\ldots,v_n)$, and $K_1=c_\alpha^{-1}g_p^{-1}(c_\alpha)$. Moreover we define $\xi_{22}^{jk}(0)$ to be the value of ξ_{22}^{jk} when $\tau=0$. It turns out that we have the following Cornish–Fisher expansion for c_α^*:

$$c_\alpha^*=c_\alpha(1+\beta_1 n^{-1})+O_p(n^{-3/2}),$$

where

$$\begin{aligned}
\beta_1=K_1\{&\tfrac{1}{2}(\xi_{22}^{jj}(0)+\xi_{11}^j\xi_{11}^j)\int_{\mathscr{D}}H_2(v_j)\,\phi(v)\,dv\\
&+(\tfrac{1}{24}\xi_4^{jjjj}+\tfrac{1}{6}\xi_{11}^j\xi_{31}^{jjj})\int_{\mathscr{D}}H_4(v_j)\,\phi(v)\,dv\\
&+(\tfrac{1}{8}\xi_4^{jjkk}+\tfrac{1}{2}\xi_{11}^j\xi_{31}^{jkk})\int_{\mathscr{D}_{j\neq k}}H_2(v_j)\,H_2(v_k)\,\phi(v)\,dv\\
&+\tfrac{1}{72}\xi_{31}^{jjj}\xi_{31}^{jjj}\int_{\mathscr{D}}H_6(v_j)\,\phi(v)\,dv\\
&+(\tfrac{1}{12}\xi_{31}^{jjk}\xi_{31}^{kkk}+\tfrac{1}{8}\xi_{31}^{jkk}\xi_{31}^{jkk})\int_{\mathscr{D}_{j\neq k}}H_2(v_j)\,H_4(v_k)\,\phi(v)\,dv\}.
\end{aligned}$$

To develop an Edgeworth expansion for the power of the bootstrap test, we define

$$F_2(x, \tau) = \int_{\mathscr{D}_\tau(x)} \{\xi_{11}^j v_j + \tfrac{1}{2}\xi_{21}^{jk}(v_j v_k - \delta^{jk}) + \tfrac{1}{6}\xi_{31}^{jkl} H_3(v_j, v_k, v_l)\}\, \phi(v)\, dv,$$

$$\begin{aligned} F_3(x, \tau) = & \int_{\mathscr{D}_\tau(x)} \{\xi_{12}^j v_j + \tfrac{1}{2}(\xi_{22}^{jk} + \xi_{11}^j \xi_{11}^k)(v_j v_k - \delta^{jk})\}\, \phi(v)\, dv \\ & + \int_{\mathscr{D}_\tau(x)} (\tfrac{1}{6}\xi_{32}^{jkl} + \tfrac{1}{2}\xi_{11}^j k_{21}^{kl})\, H_3(v_j, v_k, v_l)\, \phi(v)\, dv, \\ & + \int_{\mathscr{D}_\tau(x)} (\tfrac{1}{24}k_4^{jklm} + \tfrac{1}{6}k_{21}^{jk} k_{21}^{lm} + \tfrac{1}{6}\xi_{11}^j \xi_{31}^{klm}) \\ & \times H_4(v_j, v_k, v_l, v_m)\, \phi(v)\, dv, \\ & + \int_{\mathscr{D}_\tau(x)} \tfrac{1}{12}\xi_{21}^{jk}\xi_{31}^{lmp} H_5^{jklmp}(v_j, v_k, v_l, v_m, v_p)\, \phi(v)\, dv \\ & + \int_{\mathscr{D}_\tau(x)} \tfrac{1}{72}\xi_{31}^{jkl}\xi_{31}^{mpq} H_6^{jklmp}(v_j, v_k, v_l, v_m, v_p, v_q)\, \phi(v)\, dv. \end{aligned} \tag{3.12}$$

Now we are able to give an Edgeworth expansion for the distributions of $S^T(\tau)\, S(\tau)$ in the following theorem.

THEOREM 3.2. *Assume that X_1 has finite fifth moments and the characteristic function h_1 of X_1 satisfies the Cramér's condition, that is* $\sup_{\|t\| > b} |h_1(t)| < 1$. *Then, for any real x*

$$P\{S^T(\tau)\, S(\tau) > x\} = P\{\chi_p^2(\|\tau\|) > x\} + F_2(x, \tau)\, n^{-1/2} + F_3(x, \tau)\, n^{-1} + O(n^{-3/2}).$$

We would not give the proof of Theorem 3.2 here, since it may be derived straight forward from (3.10), (3.11), and Theorem 20.1 of Bhattacharya and Rao [3].

From Theorem 3.2 and using the delta method, we obtain the following expansion for the power of the bootstrap test ϕ_b:

$$\begin{aligned} \mathrm{pow}(\phi_b; \tau) &= P(S^T S > c_\alpha^*) \\ &= P\{\chi_p^2(\|\tau\|) > c_\alpha\} + F_2(c_\alpha, \tau)\, n^{-1/2} \\ &\quad + \{F_3(c_\alpha, \tau) - \beta_1 g_{p\tau}(c_\alpha)\}\, n^{-1} + O(n^{-3/2}). \end{aligned}$$

From (3.11) and (3.12)

$$F_2(c_\alpha; \tau) = -\int_{\mathscr{D}_\tau(c_\alpha)} \{\tfrac{1}{2}\alpha^{jkk} v_j + \tfrac{1}{2}\alpha^{jkl}\tau^l(v_j v_k - \delta^{jk}) + \tfrac{1}{3}\alpha^{jkl} H_3(v_j, v_k, v_l)\} \phi(v)\, dv. \tag{3.14}$$

From the above formula and (3.8) we see that the powers of the empirical likelihood and bootstrap tests have different second-order terms.

4. Power Comparisons

In this section we use the expansions for the powers of the empirical likelihood and bootstrap tests, developed in the previous section, to compare the powers of these two tests. Two rules are proposed for choosing practically the more powerful test, for the univariate case and the bivariate case.

From (3.7) and (3.13) we know that both ϕ_{ec} and ϕ_b have the same first order term $P\{\chi_p^2(\|\tau\|) > c_\alpha\}$ in their power functions. Thus, a comparison should be made of higher order terms. However, we shall only compare the second order terms. The reason is that when the sample size n is large enough, the difference in the power of the two tests would be dominated by the difference between the second order terms $E_2(c_\alpha; \tau)$ and $F_2(c_\alpha; \tau)$. From (3.8) and (3.14) we have

$$E_2(c_\alpha; \tau) = \int_{\mathscr{D}_\tau(c_\alpha)} \{(\tfrac{1}{3}\alpha^{jlm}\tau^l\tau^m - \tfrac{1}{6}\alpha^{jkk})\, v_j + \tfrac{1}{6}\alpha^{jkl}\tau^l(v_j v_k - \delta^{jk})\} \phi(v)\, dv,$$

$$F_2(c_\alpha; \tau) = -\int_{\mathscr{D}_\tau(c_\alpha)} \{\tfrac{1}{2}\alpha^{jkk} v_j + \tfrac{1}{2}\alpha^{jkl}\tau^l(v_j v_k - \delta^{jk}) + \tfrac{1}{3}\alpha^{jkl} H_3(v_j, v_k, v_l)\} \phi(v)\, dv.$$

4.1. *The Univariate Case*

For the univariate case (i.e. $p = 1$), put $w_1 = \sqrt{c_\alpha} - \tau$, $w_2 = \sqrt{c_\alpha} + \tau$ and $\alpha_3 = \alpha^{111}$. Then, we have from (4.1) that

$$E_2(c_\alpha; \tau) = \tfrac{1}{6}\alpha_3[(2\tau^2 - 1)\{\phi(w_1) - \phi(w_2)\} + \tau\{w_1\phi(w_1) + w_2\phi(w_2)\}],$$

$$F_2(c_\alpha; \tau) = \tfrac{1}{6}\alpha_3[-\{\phi(w_1) - \phi(w_2)\} - 3\tau\{w_1\phi(w_1) + w_2\phi(w_2)\} - 2\{w_1^2\phi(w_1) - w_2^2\phi(w_2)\}].$$

Thus, we obtain that

$$E_2(c_\alpha; \tau) - F_2(c_\alpha; \tau) = \tfrac{1}{3}\alpha_3 c_\alpha\{\phi(w_1) - \phi(w_2)\}.$$

Since $\phi(w_1)-\phi(w_2)$ is positive when $\tau>0$, and negative when $\tau<0$,

$$E_2(c_\alpha;\tau)-F_2(c_\alpha;\tau)\begin{cases}\geqslant 0, & \text{if } \alpha_3\tau>0;\\ <0, & \text{otherwise.}\end{cases} \tag{4.1}$$

From (4.1) we see that in the univariate case the relative powerfulness of the two tests depends on if the skewness parameter α_3 and τ have same sign. Since α_3 is usually unknown, we estimate it by its sample version $\hat{\alpha}_3$. Now we establish the following rule for the univariate case, which gives suggestions as to when to use the empirical likelihood test and when to use the bootstrap test.

THE UNIVARIATE RULE. *When the sample size is reasonably large, we may choose the more powerful test between the empirical likelihood and bootstrap tests by the following rule*:

$$\begin{cases}\textit{use the empirical likelihood test,} & \textit{if } \hat{\alpha}_3\tau>0;\\ \textit{use any of the two tests,} & \textit{if } \hat{\alpha}_3\tau=0;\\ \textit{use the bootstrap test,} & \textit{if } \hat{\alpha}_3\tau<0.\end{cases}$$

4.2. *The Bivariate Case*

For bivariate case of (i.e. $p=2$) write $\tau=(\tau_1,\tau_2)$ and define

$$I_1(\tau)=\int_{\mathscr{S}_\tau(x)} v_1\phi(v)\,dv,\ I_2(\tau)=\int_{\mathscr{S}_\tau(x)} v_2\phi(v)\,dv,$$

$$I_{12}(\tau)=\int_{\mathscr{S}_\tau(x)} v_1v_2\phi(v)\,dv,\ I_{11}(\tau)=\int_{\mathscr{S}_\tau(x)} (v_1^2-1)\,\phi(v)\,dv,$$

$$I_{22}(\tau)=\int_{\mathscr{S}_\tau(x)} (v_2^2-1)\,\phi(v)\,dv,\ I_{111}(\tau)=\int_{\mathscr{S}_\tau(x)} (v_1^3-3v_1)\,\phi(v)\,dv,$$

$$I_{112}(\tau)=\int_{\mathscr{S}_\tau(x)} (v_1^2-1)\,v_2\phi(v)\,dv,\ I_{122}(\tau)=\int_{\mathscr{S}_\tau(x)} v_1(v_2^2-1)\,\phi(v)\,dv,$$

$$I_{222}(\tau)=\int_{\mathscr{S}_\tau(x)} (v_2^3-3v_2)\,\phi(v)\,dv.$$

We have from (3.8) and (3.14) that

$$E_2(c_\alpha;\tau)-F_2(c_\alpha;\tau)=\alpha^{111}J_{111}(\tau)+\alpha^{112}J_{112}(\tau)+\alpha^{122}J_{122}(\tau)+\alpha^{222}J_{222}(\tau), \tag{4.2}$$

where

$$\begin{aligned} J_{111}(\tau) &= \tfrac{1}{3}(\tau_1^2+1)\,I_1(\tau) + \tfrac{2}{3}\tau_1 I_{11}(\tau) + \tfrac{1}{3}I_{111}(\tau), \\ J_{222}(\tau) &= \tfrac{1}{3}(\tau_2^2+1)\,I_2(\tau) + \tfrac{2}{3}\tau_2 I_{22}(\tau) + \tfrac{1}{3}I_{222}(\tau), \\ J_{112}(\tau) &= \tfrac{2}{3}\tau_1\tau_2 I_1(\tau) + \tfrac{1}{3}(\tau_1^2+1)\,I_2(\tau) + \tfrac{2}{3}\tau_2 I_{11}(\tau) \\ &\quad + \tfrac{4}{3}\tau_1 I_{12}(\tau) + I_{112}(\tau), \\ J_{122}(\tau) &= \tfrac{1}{3}(\tau_2^2+1)\,I_1(\tau) + \tfrac{2}{3}\tau_1\tau_2 I_2 + \tfrac{2}{3}\tau_1 I_{22}(\tau) \\ &\quad + \tfrac{4}{3}\tau_2 I_{12}(\tau) + I_{122}(\tau). \end{aligned} \tag{4.3}$$

Notice from (4.3) that $J_{111}(\tau)$, $J_{112}(\tau)$, $J_{122}(\tau)$ and $J_{222}(\tau)$ are all only dependent on τ and irrelevent to the underlined distributions. All of them can be calculated numerically for each given τ.

To find out the sign of $E_2(c_\alpha;\tau)-F_2(c_\alpha;\tau)$ given in (4.2), we also need to estimate α^{111}, α^{112}, α^{1222} and α^{222} by their sample estimators $\hat{\alpha}^{111}$, $\hat{\alpha}^{112}$, $\hat{\alpha}^{122}$ and $\hat{\alpha}^{222}$, respectively. Then we define an estimate of $E_2(c_\alpha;\tau)-F_2(c_\alpha;\tau)$, which is

$$\{E_2(c_\alpha;\tau)-F_2(c_\alpha;\tau)\}^\dagger = \hat{\alpha}^{111}J_{111}(\tau) + \hat{\alpha}^{112}J_{112}(\tau) + \hat{\alpha}^{122}J_{122}(\tau) + \hat{\alpha}^{222}J_{222}(\tau).$$

Now we are able to give the following rule for choosing a test for the bivariate case.

THE BIVARIATE RULE. *When the sample size is reasonably large, we may choose the more powerful test between the empirical likelihood and bootstrap tests by the following rule*:

$$\begin{cases} \textit{use the empirical likelihood test,} & \textit{if} \quad \{E_2(c_\alpha;\tau)-F_2(c_\alpha;\tau)\}^\dagger > 0; \\ \textit{use any of the two tests,} & \textit{if} \quad \{E_2(c_\alpha;\tau)-F_2(c_\alpha;\tau)\}^\dagger = 0; \\ \textit{use the bootstrap test,} & \textit{if} \quad \{E_2(c_\alpha;\tau)-F_2(c_\alpha;\tau)\}^\dagger < 0. \end{cases}$$

Remark. 1. For the sake of conciseness, we shall not develop rules for cases where $p \geqslant 3$ in this paper. However, one may develop some rules in the same way as for the case $p=2$ by employing general formulae for $E_2(c_\alpha;\tau)$ and $F_2(c_\alpha;\tau)$ given in (3.8) and (3.14).

5. Simulation Studies

In this section we show some simulation results to see if the theoretical rules developed in Section 4 are consistent with empirical outcomes. We considered two univariate cases and one bivariate case. The first univariate case was that where the samples were drawn from $N(0, 1)$, the standard normal distribution; we want to test $H_0: \mu = 0$ against $H_n: \mu = n^{-1/2}\tau$. In the second univariate case we drew samples from Exp(1.0), the exponential distribution with unit mean, and we tested the hypotheses $H_0: \mu = 1$ against $H_n: \mu = 1 + n^{-1/2}\tau$. In the bivariate case, we took random vectors $x_i = (x_i^1, x_i^2)$ for $i = 1, ..., n$

$$\begin{cases} x_i^1 = y_i^0 + y_i^1, \\ x_i^2 = y_i^0 + y_i^2, \end{cases}$$

where y_1^0, y_i^1, y_i^2 were drawn independently from the exponential distribution Exp(1.0). We chose sample size $n = 15$ and 30 for each of the

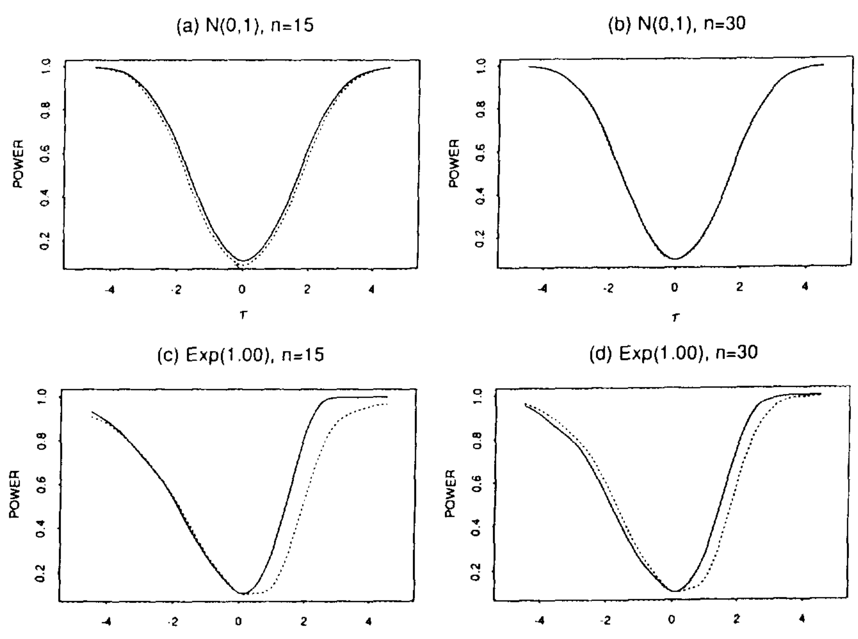

Fig. 1. The graphs depict powers curves of empirical likelihood test (solid curves) and Bootstrap test (dashed curves) as functions of τ. In cases (a) and (b) the samples were generated from $N(0, 1)$, we tested $H_0: \mu = 0$; against $H_n: \mu = n^{1/2}\tau$. In cases (c) and (d) the sample were generated from Exp(1.00), we tested $H_0: \mu = 1$; against $H_n: \mu = 1 + n^{-1/2}\tau$. The level of the test was 0.90 and the sample size $n = 15$ in (a), (c) and $n = 30$ in (b), (d).

univariate case, and $n = 30$ for the bivariate case. We fixed the level of the tests to be 0.90 in all the cases considered. The normal and exponential random variables were generated by the routines of Press *et al.* [6].

The power curves of the empirical likelihood and bootstrap test appearing in Fig. 1 were obtained by 5000 simulation at each of 19 values of τ, equally spaced within the interval $(-4.5, 4.5)$. When calculating the power of the bootstrap test, we regenerated 499 resamples for each of the 5000 simulated samples. For the bivariate case, we calculated the powers of the two tests at 225 points of $\tau = (\tau_1, \tau_2)$ within the rectangular area $(-3.5, 3.5) \times (-3.5, 3.5)$, based on 5000 simulations and 999 resamples for each simulated sample, where a contour plot of the difference between power functions of the empirical likelihood and the bootstrap tests is shown in Fig. 2.

In the first univariate case we have $\alpha_3 = 0$ since the random variables were drawn from $N(0, 1)$. According to the Univariate Rule, we can use anyone of the two tests since the powers of the tests should be very same regardless the value of τ. This is just what we see from Figs. 1(a) and 1(b). The underlying reason for this similarity is that $\alpha_3 = 0$ makes both E_2 and

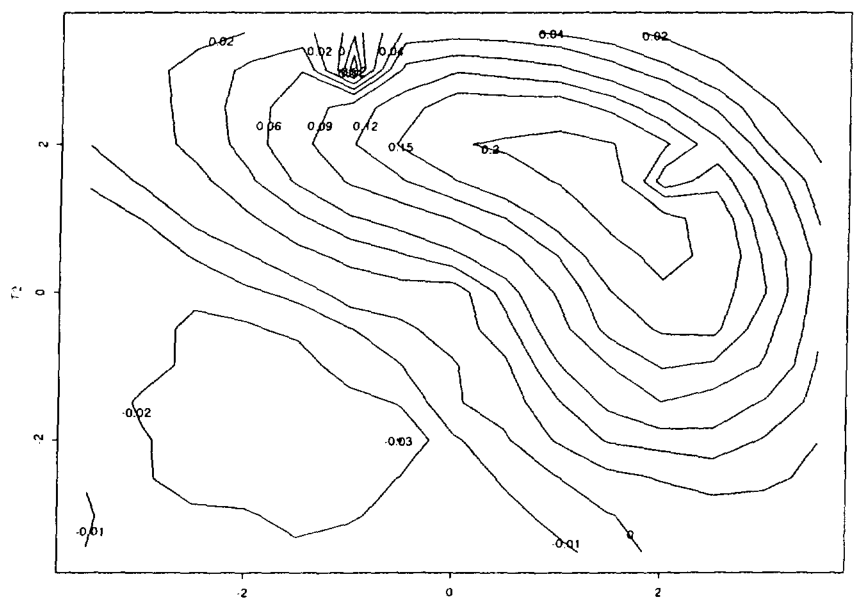

FIG. 2. Contour plot of the difference between the powers of empirical likelihood and bootstrap tests from 5,000 simulation. The random samples were $(y_i^0 + y_i^1, y_i^0 + y_i^2)$ where y_i^l $1 \leqslant l \leqslant 3$ were drawn independently from Exp(1.0). We tested $H_0: \mu = (2.2)^T$ against $H_n: \mu = (2.2)^T + n^{-1/2}\Sigma^{1/2}(\tau_1, \tau_2)^T$, where $\Sigma(1,1) = \Sigma(2,2) = 2$ and $\Sigma(1,2) = \Sigma(2,1) = 1$. The level of the test was 0.90 and the sample size $n = 30$.

F_2 vanish. Consequently, the difference between the powers of the empirical likelihood and the bootstrap tests is of order n^{-1}, rather than $n^{-1/2}$. For the second univariate case we know that $\alpha_3 = 2$. So the Univariate Rule predicts that the empirical likelihood test is more (less) powerful than the bootstrap test if $\tau > 0$ ($\tau < 0$). This is again just what Figs. 1(c) and 1(d) try to tell us. Notice that when $\tau \in (0.5, 3.5)$, the empirical likelihood test is about 20% more powerful than the bootstrap test. However, when sample size $n = 15$ which is small, we observe in the normal case that the empirical likelihood test is marginally more powerful than the bootstrap test over all range of τ. At meanwhile, in the exponential case the empirical likelihood performs similarly with the bootstrap test in the range of $\tau < 0$, where the bootstrap tests should perform better. These maybe due to the fact that the bootstrap test has to use to an explicit variance estimate, which can be very unreliable when the sample size is small. Whereas the empirical likelihood test implicity uses the true variance.

For the bivariate case, it can be shown that $\alpha^{111} = \alpha^{222} = 2(a+b)^3 + 2(a^3 + b^3)$ and $\alpha^{112} = \alpha^{122} = 2ab(a+b) + 2(a+b)^3$, where $a = 0.5(1 + 1/\sqrt{3})$ and $b = 0.5(-1 + 1/\sqrt{3})$. After numerically calculating J_{111}, J_{112}, J_{122} and J_{222}, it can be shown that

$$E_2(c_\alpha; \tau) - F_2(c_\alpha; \tau) \begin{cases} \geqslant 0 & \text{if} \quad \tau_1 + \tau_2 \geqslant 0; \\ < 0 & \text{if} \quad \tau_1 + \tau_2 < 0. \end{cases}$$

So the Bivariate Rule would suggest using the empirical likelihood test when $\tau_1 + \tau_2 > 0$, using the bootstrap test when $\tau_1 + \tau_2 < 0$, and using anyone of the two tests if $\tau_1 + \tau_2 = 0$. In Fig. 2, we give a contour plot of $\text{pow}(\phi_{ec}) - \text{pow}(\phi_b)$, which is very consistent with the prediction made from the Bivariate Rule.

ACKNOWLEDGMENTS

Thanks are particularly due to Professor Peter Hall for suggesting this research and for constant encouragement. The author thanks one referee for comments leading to improvement of the proof of Theorem 3.1.

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