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A Condition for Designing Bus-Route Type Access Site Surveys to Estimate Recreational Fishing Effort

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SUMMARY. A theoretical framework for using bus-route surveys to estimate recreational fishing effort has been established by taking into account the arrival and departure distributions of the fishing parties. Properties of a fishing effort estimator proposed by Robson and Jones (1989) are investigated. It is found that the estimator is not automatically unbiased; rather, a condition on the survey design has to be satisfied in order to be unbiased. The condition is simple and can be easily implemented.

KEY WORDS: Fishing effort estimation; Survey design; Unbiasedness.

1. Introduction

This paper is motivated by a need to develop a suitable survey methodology to estimate recreational fishing effort in Port Phillip Bay, a large protected embayment bordered by Melbourne and Geelong, the two largest cities in Victoria, Australia. The recreational fishery is very large in terms of the numbers of anglers, the time spent fishing, and the quantity of fish caught. One of the major species of fish caught is snapper. Port Phillip Bay consists of 17 major access sites for recreational fishing. The amount of fishing effort exerted by recreational fishers is one of the important indicators in the management and conservation of the recreational fisheries.

Several survey methods have been proposed in the literature for recreational fishing effort estimation. Aerial surveys and roving creel surveys have been used in the past to estimate this effort but with limited success. Roving creel surveys prove to be logistically difficult when the fishery covers a large amount of access sites over a large geographical area where aerial surveys cannot be conducted out of daylight hours. This is the case for the snapper fishery in Port Phillip Bay, where most of the fishing takes place between early evening and the early hours of the following morning.

A bus-route-type access site survey, abbreviated as bus-route survey, has been developed by Robson and Jones (1989) to overcome the problem of monitoring recreational fisheries that cover large geographical areas. A bus-route survey can cover many access sites that are usually designated boat ramps, in a large geographic area in a cost-effective manner. This type of survey has a design similar to a bus-route with prolonged stops at specified times. An agent travels along a route connecting all the access sites with predetermined schedules of arrival times, waiting periods, and departure times for each of the sites. While waiting at a site, the agent records the amount of time a fishing party's car is observed at the site.

Robson and Jones (1989) proposed an estimator for the total recreational fishing effort on New York's Great Lakes based on the observed times of fishing party cars. They derived the estimator using geometric probability of encountering fishing parties, and it was shown that the estimator is unbiased. The method used in their analysis is a typical finite sample analysis by conditioning on the fishing duration of fishing parties or, equivalently, the arrival and departure distributions of the fishing parties. Althouh the finite-sample approach provided valuable ideas for proposing the estimator, it ignored the distributions of the arrival and departure times of fishing parties. Information collected from traffic counter data in bus-route surveys conducted in Port Phillip Bay and reported in Conron and Coutin (1997) showed that distinct patterns exist in the arrival and departure times of fishing parties. The effects of these distributions on the estimator's performance and their implications on the survey design are largely unknown.

In this paper we conduct a unconditional analysis on the estimator proposed by Robson and Jones (1989) in the context of the distributions of the arrival and departure times of fishing parties. It is revealed that, regardless of the arrival and departure distributions of fishing parties, a simple condition regarding the traveling times and the waiting times must be met in to make the estimator unbiased. The condition requires that the sum of the traveling time to a site and the waiting time at the site be the same for all the sites. The condition means that, according to the existing bus-route survey design outlined in Jones et al. (1990), each site has an equal chance of being the starting site. However, it is not so for the existing bus-route survey, where the chance of being a starting site depends on the traveling time plus the waiting time for that site. The larger the traveling time plus waiting time at a site, the more likely the agent will start at that site. We believe that this type of starting rule can create bias in the fishing effort estimation as revealed in our theoretical analysis and confirmed by a simulation study reported in Chen and

Woolcock (1997). We suggest that a bus-route survey should be designed to satisfy the condition on traveling and waiting times. It has been easily implemented in the bus-route survey in Port Phillip Bay.

The paper is structured as follows. In Section 2 we establish a statistical framework for the bus-route survey and introduce some notations and definitions. The distribution of the time a fishing party is sighted by the agent is studied in Section 3. The main results of the paper are given in Section 4, which derives the condition on the survey design to ensure an unbiased estimation for the fishing effort. Section 5 presents an example from the bus-route survey in Port Phillip Bay. All proofs and derivations are given in the Appendix.

2. A Statistical Framework

Suppose that K access fishing sites are connected in a circle and are to be surveyed in a clockwise direction. Let N be the total number of independent fishing parties in the K sites, N_i of them in site i for $i = 1, \ldots, K$. Thus, $N = \sum_{i=1}^{K} N_i$.

The bus-route survey is started and finished in a time interval $[0, T]$. The sites have independent and equal probability $1/K$ to be chosen as the starting site. The remaining sites are surveyed in a clockwise direction. The starting time of the survey is determined by generating a random number in an interval $[0, \gamma]$, where $0 < \gamma < T$.

Let f_i be the bivariate probability density function of arrival and departure times of independent fishing parties at site i and f_{iS} and f_{iD} be the marginal density functions of arrival and departure times, respectively, for $i = 1, \ldots, K$.

Let $\mu_{iS} = \int_0^T a f_{iS}(a)da$ and $\mu_{iD} = \int_0^T b f_{iD}(b)db$ be the mean arrival and finishing times of the fishing parties at site i. Then the mean fishing time by each party at site i is $\mu_{iD} - \mu_{iS}$, and the total average fishing effort is

$$T_{fe} = \sum_{i=1}^{K} N_i \times (\mu_{iD} - \mu_{iS}).$$

The aim of the survey is to estimate T_{fe}.

Let us define, for $i = 1, \ldots, K$ and $j = 1, \ldots, N_i$, the following:

- d_i = the agent's travel time between site $i-1$ to site i, where site 0 = site K;
- w_i = the agent's waiting time at site i;
- a_i = the arrival time of the agent at site i;
- S_{ij} = the arrival time of the jth fishing party at site i
- D_{ij} = the departure time of the jth fishing party at site i;
- X_{ij} = the length of time the jth fishing party is sighted at site i.

We assume throughout this paper that the joint density distributions of the arrival and departure times satisfy the following conditions, i.e., for each $i = 1, \cdots K$,

$$f_i(S, D) = 0 \quad \text{for } D - S < \max\{w_i\}$$
$$f_{iS}(S) = 0 \quad \text{for } S > T - \min\{w_i\}$$

and

$$f_{iD}(D) = 0 \quad \text{for } D < \max\{w_i\}. \tag{2.1}$$

These conditions mean that the minimum fishing time must be larger than the maximum waiting time and that no arrival occurs toward the end of the fishing day and no departure at the beginning of the fishing day, respectively. These are very trivial survey conditions and can be easily implemented.

An estimator for T_{fe} proposed by Robson and Jones (1989) is given by

$$\hat{T}_{fe} = T \sum_{i=1}^{K} \sum_{j=1}^{n_i} \frac{X_{ij}}{w_i} \tag{2.2}$$

where n_i is the number of sightings at site i. The aim of this paper is to find conditions to ensure that $\hat{T}_{fe}$ is an unbiased estimator for T_{fe}.

3. Distribution of X_{ij}

We see from (2.2) that it is crucial to know the distribution of X_{ij}, which is the time the jth fishing party is sighted at site i by the agent, in the investigation of $\hat{T}_{fe}$. We first look at the conditional distributions of a_i, the arrival time of the agent at site i, given that the survey is started at various sites. Let $U[a, b]$ be a uniform distribution in $[a, b]$, and $a_{i|k}$ denote a_i conditional on starting the survey at site k. Then

$$a_{i|i} \sim U[0, \gamma],$$
$$a_{i|i-1} \sim U[w_{i-1} + d_i, w_{i-1} + d_i + \gamma],$$
$$a_{i|1} \sim U\left[\sum_{j=1}^{i-1} w_j + \sum_{j=2}^{i} d_j, \sum_{j=1}^{i-1} w_j + \sum_{j=2}^{i} d_j + \gamma\right],$$
$$a_{i|K} \sim U\left[\sum_{j=1}^{i-1} w_j + \sum_{j=1}^{i} d_j + w_K, \sum_{j=1}^{i-1} w_j + \sum_{j=1}^{i} d_j + w_K + \gamma\right],$$

and

$$a_{i|i+1} \sim U\left[\sum_{j\neq i}^{i-1} w_j + \sum_{j\neq i+1}^{i} d_j, \sum_{j\neq i}^{i-1} w_j + \sum_{j\neq i+1}^{i} d_j + \gamma\right],$$

where $\sim$ means distributed. Denote $a_{i|k} \sim U[t_{ik}^l, t_{ik}^r]$. Then, by summarizing these expressions, we have

$$t_{ik}^l = \begin{cases} \sum_{j=k}^{i-1} w_j + \sum_{j=k+1}^{i} d_j & \text{if } i \geq k, \\ \sum_{j=1}^{i-1} w_j + \sum_{j=1}^{i} d_j \\ \quad + \sum_{j=k}^{K} w_j + \sum_{j=k+1}^{K} d_j & \text{if } i < k. \end{cases} \tag{3.1}$$

and

$$t_{ik}^r = t_{ik}^l + \gamma. \tag{3.2}$$

Here we use a convention that $\sum_{j=i}^{i-1} w_j = \sum_{j=1+i}^{i} d_j = 0$.

Note that t_{ik}^l and t_{ik}^r are completely determined by the survey design, i.e., by the waiting times w_i, the travel times between sites d_i and γ.

In the following we investigate the distribution of X_{ij}. Three cases for the value of X_{ij} are as follows:

$$X_{ij} = \begin{cases} 0 & \text{if } (a_i, a_i + w_i) \cap (S_{ij}, D_{ij}) = \emptyset; \\ x \in (0, w_i) & \text{if } S_{ij} = a_i + w_i - x \text{ or } D_{ij} = a_i + x; \\ w_i & \text{if } (a_i, a_i + w_i) \subseteq (S_{ij}, D_{ij}). \end{cases} \tag{3.3}$$

Note that (S_{ij}, D_{ij}) is the time interval when the party is fishing and $(a_i, a_i + w_i)$ is the time interval when the agent is at the site. Thus, X_{ij} is a mixture of discrete and continuous distributions.

LEMMA. *Under the conditions in (2.1), the conditional distribution of X_{ij} given that the survey is started at site k is the following:*

$$\begin{aligned} &Pr(X_{ij} = 0 \mid \textit{start at site } k) \\ &= \int_{t^r_{ik}+w_i}^{T} f_{iS}(S)\,dS + \int_{t^l_{ik}+w_i}^{t^r_{ik}+w_i} \frac{S - w_i - t^l_{ik}}{\gamma} f_{iS}(S)\,dS \\ &\quad + \int_0^{t^l_{ik}} f_{iD}(D)\,dD + \int_{t^l_{ik}}^{t^r_{ik}} \frac{t^r_{ik} - D}{\gamma} f_{iD}(D)\,dD, \quad (3.4) \end{aligned}$$

$$\begin{aligned} &Pr(X_{ij} = w_i \mid \textit{start at site } k) \\ &= \int_0^{t^r_{ik}} f_{iS}(S)\,dS + \int_{t^l_{ik}}^{t^r_{ik}} \frac{t^l_{ik} - S}{\gamma} f_{iS}(S)\,dS \\ &\quad - \int_0^{t^r_{ik}+w_i} f_{iD}(D)\,dD \\ &\quad + \int_{t^l_{ik}+w_i}^{t^r_{ik}+w_i} \frac{D - w_i - t^l_{ik}}{\gamma} f_{iD}(D)\,dD \quad (3.5) \end{aligned}$$

and

$$\begin{aligned} &Pr(0 < X_{ij} < x \mid \textit{start at site } k) \\ &= \int_{t^l_{ik}+w_i-x}^{t^l_{ik}+w_i} \left\{ \frac{S + x - w_i - t^l_{ik}}{\gamma} f_{iS}(S)\,dS \right. \\ &\qquad \left. + \frac{D + x - w_i - t^l_{ik}}{\gamma} f_{iD}(D)\,dD \right\} \\ &\quad + \int_{t^r_{ik}+w_i-x}^{t^r_{ik}+w_i} \left\{ \frac{t^r_{ik} - S + w_i}{\gamma} f_{iS}(S)\,dS \right. \\ &\qquad \left. + \frac{t^r_{ik} - D + w_i}{\gamma} f_{iD}(D)\,dD \right\} \\ &\quad + \int_{t^l_{ik}+w_i}^{t^r_{ik}+w_i-x} \frac{x}{\gamma} \left\{ f_{iS}(S)\,dS + f_{iS}(D)\,dD \right\}. \quad (3.6) \end{aligned}$$

The proof of the lemma is deferred to the Appendix.

Taking the derivative with respect to x in (3.6), the conditional density function of X_{ij} within $(0, w_i)$ is

$$f_{X_{ij|k}}(x) = \frac{1}{\gamma} \int_{t^l_{ik}+w_i-x}^{t^r_{ik}+w_i-x} \left\{ f_{iS}(S)\,dS + f_{iD}(D)\,dD \right\}. \quad (3.7)$$

The unconditional distribution of X_{ij} is just the average of the conditional ones, as given in (3.4),(3.5), and (3.6), over k as Pr(start at site k) $=1/K$ and thus can be expressed easily. In particular

$$\begin{aligned} &Pr(X_{ij} = w_i) \\ &= \frac{1}{K} \sum_{k=1}^{K} \left\{ \int_{t^l_{ik}+w_i}^{t^r_{ik}+w_i} \frac{D - w_i}{\gamma} f_{iD}(D)\,dD \right. \\ &\quad - \int_{t^l_{ik}}^{t^r_{ik}} \frac{S}{\gamma} f_{iS}(S)\,dS + \int_0^{t^r_{ik}} f_{iS}(S)\,dS \\ &\quad + \frac{t^l_{ik}}{\gamma} \int_{t^l_{ik}}^{t^r_{ik}} f_{iS}(S)\,dS - \int_0^{t^r_{ik}+w_i} f_{iD}(D)\,dD \\ &\quad \left. - \frac{t^l_{ik}}{\gamma} \int_{t^l_{ik}+w_i}^{t^r_{ik}+w_i} f_{iD}(D)\,dD \right\}. \end{aligned}$$

4. A Condition for an Unbiased Estimator

In this section we propose a condition for the survey design such that $\hat{T}_{fe}$ is an unbiased estimator of T_{fe}. The condition is

$$d_1 + w_k = d_2 + w_1 = \cdots = d_k + w_{k-1} = \gamma. \quad (4.1)$$

This condition means that the sum of each individual waiting time at a site and traveling time to that site is the same and equal to γ. The condition (4.1) implies only that their sums be the same, not that all waiting times and traveling times be the same. Notice that the survey is started uniformly within $[0, \gamma]$. In practice γ is calculated by dividing the length of the fishing day by the number of access sites, i.e., $\gamma = T/K$. As the traveling times d_i between access sites is predetermined, the waiting time at each access site is given by $w_i = \gamma - d_i$.

From (3.1) and (3.2) we see that the previous condition means that, for any $i \in \{1, 2, \ldots, k\}$,

$$U^K_{k=1}[t^l_{ik}, t^r_{ik}) = [0, T - w_i - d_{i+1}]$$

and

$$[t^l_{ik'}, t^r_{ik'}) \cap [t^l_{ik''}, t^r_{ik''}) = \emptyset \qquad \text{if } k' \neq k'', \quad (4.2)$$

where $\emptyset$ is the empty set. The condition leaves no gaps and overlaps in the arrival time of the agent at site i regardless where the survey is started. Thus, it allows the agent to cover the entire distribution of the fishing duration without gaps and overlaps.

The main result of the paper is given in the following theorem, whose proof is deferred to the Appendix.

THEOREM. *For any joint arrival and departure distributions f_i, if they obey the conditions in* (2.1) *and at the same time* (4.1) *is satisfied, then $\hat{T}_{fe}$ is an unbiased estimator of T_{fe}.*

As explained in Section 2, the conditions in (2.1) are very trivial conditions. It can be seen from the proof that the condition (4.1) plays a crucial role in obtaining the unbiasedness of $\hat{T}_{fe}$. We have not been able to show vigorously that the condition (4.1) is also necessary. However, if the condition is not satisfied, gaps and overlaps will exist among the intervals $[t^l_{ik}, t^r_{ik})$. By looking at the integrals used in the proof, there might be some arrival and departure distributions f_i such that $\hat{T}_{fe}$ is biased.

Robson and Jones (1989) showed that $\hat{T}_{fe}$ is a Horvitz-Thompson type estimator, which led to $\hat{T}_{fe}$ being an unbiased estimator for T_{fe}. However, it is only so by conditioning on the arrival and departure distributions. Unconditionally, $\hat{T}_{fe}$ is not a Horvitz-Thompson-type estimator, as w_i/T is not the probability of observing fishing parties. Thus, the unbiasedness of $\hat{T}_{fe}$ is not automatic and requires condition (4.1) when designing the survey.

As described in Robson and Jones(1989), difficulties exist in estimating the variation of $\hat{T}_{fe}$ because only one agent is available per route per day, resulting in a lack of replication in the observations for the variance estimation. When the survey is conducted over L days, a conservative estimate for the variance is

$$S^2 = \{L(L-1)\}^{-1} \sum_{l=1}^{L} (\hat{T}_{fe}^{l} - \bar{\hat{T}}_{fe})^2,$$

where $\hat{T}_{fe}^{l}$ is the estimate for the lth day and

$$\bar{\hat{T}}_{fe} = L^{-1} \sum_{l=1}^{L} \hat{T}_{fe}^{l}$$

is the average fishing effort estimate over the L days. Details on this conservative variance estimate and the related issue are available in Robson and Jones (1989). More research is certainly needed in this area.

5. An Example

Bus-route surveys for recreational fishing effort have been conducted in Port Phillip Bay 17 major access sites. The surveys typically covered the months of November to March of the following year. This is the snapper season in Port Phillip Bay.

In the surveys conducted in the seasons of 1994–1995 to 1995–1996, the 17 sites were divided into two groups. The Mornington Peninsula, the eastern side of the bay, consisted of nine sites, and the Bellarine Peninsula, the western side of the bay, consisted of eight sites. Each group was assigned a route and was surveyed in a fishing day. The design of Jones et al. (1990) was adopted. Table 2 ($w_i = 30$) presents a survey design used for the nine sites in the eastern side of the bay, with the site names extracted. The table contains the traveling time between the sites and the waiting time at each site. Obviously, the condition (4.1) was not satisfied.

The new survey design proposed in this paper has been implemented in the 1996–1997 survey, i.e., the traveling and the waiting times satisfied (4.1). Using this opportunity of change, the access sites has been classified into three groups according to the three geographical clusters around the bay—Mornington Peninsula, Melbourne, and Bellarine Peninsula—consisting of five, four, and six sites, respectively. Two sites from the previous survey were considered insignificant and removed. Table 1 gives a design for a group of five sites in the Mornington Peninsula. The agent started the survey at Patterson River and finished at Frankston. The sums of traveling time and the waiting time were 96 minutes for all the sites. The table also gives the observed number of fishing parties n_i, the total amount of time observed $\sum_{j=1}^{n_i} X_{ij}$, and the estimated fishing effort at each site. The estimated total fishing effort for the day was 44,110 minutes, or 735.17 hours.

Table 1

One-day data set from a bus-route survey conducted in Port Philip Bay in November 1996. The travel time to an access site and the waiting time spent by the agent is denoted by d_i and w_i, respectively. The number of observations is denoted by n and the total observed fishing party hours by $\sum X_i$. The estimated effort as given in (2.2) is $\hat{T}_{fe}$.

Access site	d_{i-1}	w_i	n_i	$\sum_{j=1}^{n_i} X_{ij}$	$\hat{T}_{fe}$
Patterson River	26	70	23	4185	28,697
Morinington	51	45	5	225	2400
Rye	51	45	7	315	3360
Safety Beach	36	60	1	60	480
Frankston	51	45	23	860	9173

Table 2

Travel times between access sites, d_i, for the old design with two different waiting times, $w_i = 30$ and $w_i = 40$. The access sites are subscripted by i.

Access site	d_i	
(i)	$w_i = 30$	$w_i = 40$
1	24	14
2	14	14
3	20	10
4	17	7
5	15	8
6	19	9
7	33	20
8	16	6
9	22	12

6. Conclusion

This paper proposed condition (4.1), which ensures that the estimator for the total fishing effort is unbiased without assuming any knowledge of the arrival and departure distributions of the fishing parties. The condition ensures that the survey time covers the entire range of the arrival and departure distributions of the fishing parties without gaps and overlaps.

The survey condition (4.1) is very easy to implement in practice by adjusting the traveling and waiting times. However, in the survey design a minimum level of waiting time has to be maintained to capture enough information. When there is a need to increase the waiting time to increase the sample size, the traveling times must be reduced. However, if logistic difficulties exist in reducing the traveling time, another survey agent should be sought to maintain conditions (4.1) and realistic traveling times at the same time.

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RÉSUMÉ

On a établi un contexte théorique pour l'utilisation d'enquêtes sur les trajets de car, pour estimer l'effort relatif à la pèche d'agrément en prenant en compte les distributions de départ et d'arrivée des parties de pèche. Les propriétés de l'estimateur de l'effort de pèche proposé par Robson et Jones (1989) sont étudiées. On montre que cet estimateur n'est pas systématiquement sans biais, mais qu'une condition portant sur le

schéma de l'enquête doit être vérifiée pour que l'estimateur soit sans biais. Cette condition est simple et peut être aisément remplie.

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Appendix

Proof of Lemma

We show only the derivation for the conditional probability $X_{ij} = w_i$ as the other ones can be derived in a similar manner. Notice that

$$
\begin{aligned}
&Pr(X_{ij} = w_i \mid \text{start at site } k) \\
&\quad = Pr(S_{ij} < a_i < D_{ij} - w_i \mid a_i \sim U[t_{ik}^l, t_{ik}^r]) \\
&\quad = \iint \int_{S<a<D-w_i, S<D-w_i} f_i(S,D) u(a)\, da\, dS\, dD,
\end{aligned}
$$

where u is the density function of $U[t_{ik}^l, t_{ik}^r]$.

As

$$
\int_{S<a<D-w_i} u(a) da
= \begin{cases}
0 & \text{if either } D - w_i < t_{ik}^l \text{ or } S > t_{ik}^r \text{ ;} \\
1 & \text{if } S < t_{ik}^l \text{ and } t_{ik}^r < D - w_i \text{ ;} \\
\dfrac{D - w_i - t_{ik}^l}{\gamma} & \text{if } S < t_{ik}^l \text{ and } t_{ik}^l < D - w_i < t_{ik}^r \text{ ;} \\
\dfrac{t_{ik}^r - S}{\gamma} & \text{if } t_{ik}^l < S < t_{ik}^r \text{ and } D - w_i > t_{ik}^r \text{ ;} \\
\dfrac{D - w_i - S}{\gamma} & \text{if } t_{ik}^l < S \text{ and } D - w_i < t_{ik}^r,
\end{cases}
$$

then

$$
\begin{aligned}
&Pr(X_{ij} = w_i \mid \text{ start at site } k) \\
&\quad = \iint_{S<t_{ik}^l, D>t_{ik}^r+w_i} f_i(S,D)\, dS\, dD \\
&\qquad + \iint_{S<t_{ik}^l, t_{ik}^l+w_i<D<t_{ik}^r+w_i} \frac{D - w_i - t_{ik}^l}{\gamma} f_i(S,D)\, dS\, dD \\
&\qquad + \iint_{t_{ik}^l<S<t_{ik}^r, D>t_{ik}^r+w_i} \frac{t_{ik}^r - S}{\gamma} f_i(S,D)\, dS\, dD \\
&\qquad + \iint_{t_{ik}^l<S<D, D<t_{ik}^r+w_i} \frac{D - w_i - S}{\gamma} f_i(S,D)\, dS\, dD \\
&\quad = \left\{ \iint_{S<t_{ik}^l, D>t_{ik}^r+w_i} + \iint_{t_{ik}^l<S<t_{ik}^r, D>t_{ik}^r+w_i} \right\} \\
&\qquad \times f_i(S,D)\, dS\, dD \\
&\qquad + \left\{ \iint_{S<t_{ik}^l, t_{ik}^l+w_i<D<t_{ik}^r+w_i} + \iint_{t_{ik}^l<S<D, D>t_{ik}^r+w_i} \right\} \\
&\qquad \times \frac{D - w_i}{\gamma} f_i(S,D)\, dS\, dD \\
&\qquad + \left\{ \iint_{t_{ik}^l<S<t_{ik}^r, D>t_{ik}^r+w_i} - \iint_{S<t_{ik}^l, t_{ik}^l+w_i<D<t_{ik}^r+w_i} \right\} \\
&\qquad \times \frac{t_{ik}^l}{\gamma} f_i(S,D)\, dS\, dD \\
&\qquad - \left\{ \iint_{t_{ik}^l<S<D, D>t_{ik}^r+w_i} + \iint_{t_{ik}^l<S<t_{ik}^r, D>t_{ik}^r+w_i} \right\} \\
&\qquad \times \frac{S}{\gamma} f_i(S,D)\, dS\, dD \\
&\quad = \int_{S<t_{ik}^r} f_{iS}(S)\, dS - \int_{D<t_{ik}^r+w_i} f_{iD}(D)\, dD \\
&\qquad + \int_{t_{ik}^l+w_i<D<t_{ik}^r+w_i} \frac{D - w_i - t_{ik}^l}{\gamma} f_{iD}(D)\, dD \\
&\qquad + \int_{t_{ik}^l<S<t_{ik}^r} \frac{t_{ik}^l - S}{\gamma} f_{iS}(S)\, dS.
\end{aligned}
$$

Proof of Theorem

We first work out $E(X_{ij})$, which is

$$
\begin{aligned}
E(X_{ij}) &= \frac{1}{K} \sum_{k=1}^{K} E(X_{ij} \mid \text{start at site } k) \\
&= \frac{1}{K} \sum_{k=1}^{K} \left\{ \int_0^{w_i} x f_{X_{ij}}(x)\, dx \right. \\
&\qquad \left. + w_i P(X_{ij} = w_i \mid \text{start at site } k) \right\}.
\end{aligned}
\tag{A.1}
$$

As condition (4.1) implies (4.2), we have from (3.7) that

$$
\begin{aligned}
&\frac{1}{K} \sum_{k=1}^{K} \int_0^{w_i} x f_{X_{ij|k}}(x)\, dx \\
&\quad = \frac{1}{\gamma K} \int_0^{w_i} x\, dx \\
&\qquad \times \sum_{k=1}^{K} \int_{t_{ik}^l+w_i-x}^{t_{ik}^r+w_i-x} \left\{ f_{iS}(S)\, dS + f_{iD}(D)\, dD \right\} \\
&\quad = \frac{1}{\gamma K} \int_0^{w_i} x\, dx \int_{w_i-x}^{T+w_i-x} \left\{ f_{iS}(S)\, dS + f_{iD}(D)\, dD \right\}
\end{aligned}
$$

$$= \frac{1}{\gamma K}\int_0^{w_i} 2x\,dx = \frac{{w_i}^2}{\gamma K}. \tag{A2}$$

Note that condition (4.1) implies that $t_{ik}^l = (k-1)\gamma$, $t_{ik}^r = k\gamma$, and $T = \gamma K$.

From (3.5)

$$\frac{1}{K}\sum_{k=1}^{K} P(X_{ij} = w_i \mid \text{start at site } k)$$
$$= \frac{1}{K}\sum_{k=1}^{K}\Bigg\{\int_0^{k\gamma} f_{iS}(S)\,dS + (k-1)\int_{(k-1)\gamma}^{k\gamma} f_{iS}(S)\,dS - \int_0^{k\gamma+w_i} f_{iD}(D)\,dD - (k-1)\int_{(k-1)\gamma+w_i}^{k\gamma+w_i} f_{iD}(D)\,dD + \int_{w_i}^{T-d_{i+1}} \frac{(D-w_i)}{\gamma} f_{iD}(D)\,dD - \int_0^{T-w_i-d_{i+1}} \frac{S}{\gamma} f_{iS}(S)\,dS\Bigg\}.$$

It may be readily shown that

$$\frac{1}{K}\sum_{k=1}^{K}\Bigg\{\int_{k\gamma}^{T_i} f_{iS}(S)\,dS - (k-1)\int_{(k-1)\gamma}^{k\gamma} f_{iS}(S)\,dS\Bigg\}$$
$$= \frac{1}{K}\sum_{k=1}^{K}\Bigg\{\int_{k\gamma+w_i}^{T_i} f_{iD}(D)\,dD - (k-1)\int_{(k-1)\gamma+w_i}^{k\gamma+w_i} f_{iD}(D)\,dD\Bigg\} = 0.$$

Thus,

$$\frac{1}{K}\sum_{k=1}^{K} w_i P(X_{ij} = w_i \mid \text{start at site } k) = \frac{w_i}{\gamma K}\{(\mu_{iD} - \mu_{iS}) - w_i\}. \tag{A.3}$$

Substituting (A.2) and (A.3) into (A.1), we have

$$E(X_{ij}) = \frac{w_i}{T}\{(\mu_{iD} - \mu_{iS}) - w_i\} + \frac{{w_i}^2}{T} = w_i\frac{(\mu_{iD} - \mu_{iS})}{T}. \tag{A.4}$$

Thus,

$$E(\hat{T}_{fe}) = E\Bigg\{T\sum_{k=1}^{K}\sum_{j=1}^{N_k}\frac{X_{kj}}{w_k}\Bigg\} = \sum_{k=1}^{K}\sum_{j=1}^{N_k}(\mu_{iD} - \mu_{iS}) = T_{fe}. \tag{A.5}$$