

Physics-guided Statistical Data Fusion for Reconstructing 3D Current Fields of Oceanic Eddies

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Abstract

Accurate reconstruction of three-dimensional ocean current fields is critical for understanding ocean dynamics and real-time conduct of modern oceanographic field campaigns, particularly for mesoscale eddy surveys. We propose a physics-guided modeling and learning framework for multi-source data fusion to estimate the three-dimensional (3D) current structure of oceanic eddies by integrating satellite altimetry and temperature data, ocean reanalysis data, and in situ drifting buoy observations. The approach leverages geostrophic balance, derived from the Navier-Stokes equations, to guide a neural network trained on GLORYS reanalysis data in inferring subsurface currents from surface conditions. The surface conditions were estimated using a high-dimensional linear mixed model, which integrates systematically biased satellite altimetry and sparse drifting buoy data, allowing for spatially adaptive bias correction and yielding more accurate and spatially coherent surface velocity fields. This framework was deployed in a September 2024 field campaign targeting a cyclonic eddy in the Kuroshio Extension, guiding a real-time control of seven underwater gliders. Compared with existing data products, our method demonstrated substantially improved accuracy in cross-validation with drifting buoys and stronger consistency with ADCP observations. The resulting glider trajectories provided enhanced spatial coverage of the eddy interior, enabling the first successful high-resolution controlled network survey of a mesoscale eddy.

Keywords: bias correction, data fusion, neural networks, oceanic eddy, physics-guided modeling.

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1 Introduction

Mesoscale eddies play a central role in ocean circulation, mediating the transport of heat, salt and biogeochemical tracers (McGillicuddy et al., 1998; Jing et al., 2020). Understanding their three-dimensional (3D) structure is critical for advancing physical oceanography and improving ocean-climate predictions. Despite progress in satellite remote sensing and numerical models (Archer et al., 2025), high-resolution in situ measurements of real-world mesoscale eddies remain limited, particularly in regions of intense eddy activity, where conducting controlled surveys is challenging due to strong currents.

The problem of reconstructing 3D currents of ocean eddies is motivated by a dedicated field campaign in September 2024 that targeted a cyclonic eddy in the Kuroshio Extension, a region of high eddy kinetic energy and strong currents (Figure 1(a)–(b)). The goal of the field campaign was to use a network of underwater gliders to measure the hydrographic features of the eddy. The target eddy exhibited surface currents exceeding 1 m/s, far stronger than the cruising speed (approximately 0.3 m/s) of underwater gliders (Wang et al., 2022), creating substantial challenges for controlled sampling. Accurate estimation of the eddy velocity field is crucial for effective survey planning, pathway design, and real-time control of underwater gliders (Lin et al., 2024; Su et al., 2026).

Large-scale ocean numerical models powered by partial differential equations such as the Navier-Stokes equations have been well developed to describe the physical dynamics of the ocean (Vallis, 2017). Those models are physics knowledge-driven, which may deviate from real measurements. There has been much effort to integrate physical models and real observations, called data assimilation (Evensen, 2009), which produces reanalysis data for the ocean and atmosphere. GLORYS (Lellouche et al., 2018), developed by Mercator Ocean International, is one of the publicly available high-quality reanalysis data

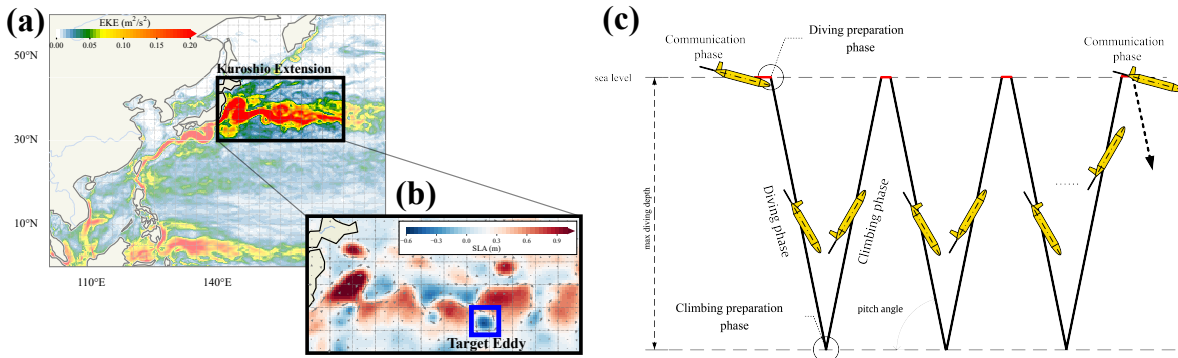


Figure 1: (a) Average eddy kinetic energy (EKE) across the northwestern Pacific in 2023, derived from AVISO sea surface geostrophic currents (Pascual et al., 2006). (b) Sea level anomaly and geostrophic velocity field in the Kuroshio Extension on September 12, 2024 from AVISO. (c) Glider diving and climbing motions in the vertical section.

sets. Embedding physical laws into learning has also gained increasing attention in machine learning, particularly through the use of physics-informed neural networks (PINNs) to solve partial differential equations (PDEs) and estimate parameters in PDEs (Raissi et al., 2019). Parameter estimation for ordinary differential equations is also considered in Liang and Wu (2008) and Tan et al. (2024). All those methods require observed variables in the physical system. For the target eddy, underwater in situ observations are unavailable for model training, motivating an approach that learns the relationship between surface and interior currents from reanalysis data.

Covariance modeling plays a central role in describing spatial dependence and Kriging interpolation. Mixed-effect models with parametric covariance structures are commonly used in spatial statistics (Lan and Bao, 2024; Song et al., 2024) and data fusion of multi-source observations (Bao et al., 2023). The Matérn class of covariances is usually applied to data with spatially stationary covariances that only depend on the distance between two locations and decay as the distance increases. Particularly, Furrer et al. (2006) developed

tapered Matérn Kriging, and [Cressie and Johannesson \(2008\)](#) proposed fixed rank Kriging. [Song et al. \(2024\)](#) developed a spatial linear mixed model with spherical harmonic transformation for spatial covariances. However, reliance on parametric covariance models can be restrictive when the underlying spatial process is anisotropic or nonstationary. Note that the geometric structure of an eddy may not be symmetric, and the velocity on the edge of an eddy is much stronger than at its center. This indicates that the dependence on the edge is stronger and the spatial covariance of velocities is nonstationary, implying the parametric approaches for covariances may not be appropriate for estimating the eddy velocity field.

To estimate the three-dimensional current field of mesoscale eddies, we develop a physics-guided statistical learning framework that combines physical constraints from geostrophic balance, and multi-source observational fusion via a high-dimensional linear mixed model with a multi-directional bandable covariance structure. The proposed method can consistently recover the surface velocity field by integrating the satellite data that are subject to bias but with good spatial coverage and the drifting buoy data which are more accurate but sparsely observed. Based on the estimated surface velocity field and a physics-guided statistical model, we predict the interior velocities via a deep learning approach trained on the GLORYS reanalysis data. Using the estimated velocity field, we guided real-time control of seven gliders, enabling the first controlled survey of a mesoscale eddy and yielding valuable in situ data. Post-mission analysis further revealed that our reconstructed fields reduced average navigation errors to less than half of those obtained using GLORYS reanalysis data, substantially enhancing the quality of glider-based data collection.

Beyond the successful field campaign, this study makes two main contributions in statistical methodology for geoscience. First, to overcome the challenge of no underwater

observations, we propose to learn a statistical relationship between the ocean interior and surface fields, using a deep learning model trained on reanalysis data, guided by the physical relationship of an eddy under the geostrophic balance approximation of the Navier-Stokes equations (Vallis, 2017). Second, we develop a high-dimensional linear mixed model for integrating different sources of data with spatial nonstationary covariance matrices. We introduce a framework to estimate multi-directional bandable covariance matrices by applying spatial banding to temporally kernel-weighted sample covariance matrices, which is broader than the uni-directional bandable estimation considered in Bickel and Levina (2008) and Cai et al. (2010). While demonstrated in a dedicated field campaign, the proposed methods are applicable to other dynamic systems where partial physical knowledge and sparse observations need to be combined for reliable inference and prediction.

The remainder of the paper is organized as follows. Section 2 introduces the background of the study, geostrophic models, and the multi-source data. Section 3 estimates surface velocities by a linear mixed model that integrates multi-source surface observations. Section 4 proposes the physics-guided statistical method for estimating the eddy’s interior velocity. Section 5 reports the application of the proposed approach in tracking and surveying a real-world eddy. Theoretical justification of proposed methods, additional technical and numerical details, and simulation studies are relegated to the Supplementary Material (SM).

2 Scientific problem, physical model and data

We first formulate the scientific problems that link ocean current field reconstruction with underwater glider control. Let $\mathcal{D} = \mathcal{D}_x \times \mathcal{D}_y \times \mathcal{D}_z$ represent the spatial domain of the target eddy, where the horizontal domain spans from 158°E to 161°E in longitude and from 32°N to 35°N in latitude, while the vertical domain extends from the sea surface to a depth of

5727.9 meters. We aim to estimate the daily-averaged current field of the target eddy and use it to control the headings of the gliders. We consider a spatial resolution of $1/12^\circ$ and 50 layers in depth, which results in a high-dimensional (HD) set of grids. The study area is located in the mid-latitudes, where the daily-averaged oceanic velocity field is primarily governed by geostrophic balance (Vallis, 2017), as the Earth’s rotation plays the major role in ocean dynamics at these latitudes. Geostrophic balance occurs when the Coriolis force, arising from the Earth’s rotation, exactly counteracts the seawater pressure-gradient force, which holds not only for large-scale background currents but also within mesoscale eddies. High-frequency non-geostrophic components, such as Ekman flows induced by wind and tidal motions, are much weaker than the geostrophic part and are substantially reduced after daily averaging. On average, the ageostrophic kinetic energy accounts for only 2.55% of the total kinetic energy in the study region as shown by GLORYS daily data, which will be introduced in Section 2.2; see an analysis in Section S1.1 of the SM for details. Thus, we focus on the geostrophic current field.

Let $\tilde{u}_t(x, y, z)$ and $\tilde{v}_t(x, y, z)$ be the true geostrophic zonal (eastward) and meridional (northward) velocities, respectively, at time t , longitude x , latitude y , and depth z , for $(x, y, z) \in \mathcal{D}$. The eddy center at time t is the location of the local minimum in the sea level anomaly (SLA) field for a cyclonic eddy, which can be identified from satellite-derived SLA products (e.g., AVISO; see Section 2.2), as demonstrated by Chelton et al. (2011). Denote the coordinates of the eddy center by (x_t^c, y_t^c) and its translational velocities by $u_t^{(c)}$ and $v_t^{(c)}$. The velocity field in the eddy-centered coordinate system is defined as

$$u_t(x, y, z) = \tilde{u}_t(x + x_t^c, y + y_t^c, z) - u_t^{(c)} \quad \text{and} \quad v_t(x, y, z) = \tilde{v}_t(x + x_t^c, y + y_t^c, z) - v_t^{(c)}, \quad (2.1)$$

where (x, y) now represent the zonal and meridional displacements relative to the eddy center. The translational velocities $u_t^{(c)}$ and $v_t^{(c)}$ can be estimated from the displacement of

the eddy center between consecutive days.

A glider performs continuous dive-and-climb movements to measure ocean temperature and salinity (Figure 1(c)). Each dive descends to a depth of $h = 1,000$ meters along a prescribed heading. Once submerged, the glider's position and speed are unavailable until it surfaces again. Upon surfacing, a new heading can be instructed to the glider for the next dive. To keep the glider on a designed pathway, it is required to know the eddy velocities $u_t(x, y, z)$ and $v_t(x, y, z)$. The glider's total velocity relative to the eddy center for the k th dive on day t is $u_{t,k}^{(g)}(x, y, z) = u_t(x, y, z) + u_{t,k}^{(m)}(x, y)$ and $v_{t,k}^{(g)}(x, y, z) = v_t(x, y, z) + v_{t,k}^{(m)}(x, y)$, where $u_{t,k}^{(m)}(x, y)$ and $v_{t,k}^{(m)}(x, y)$ denote the glider's self-propulsion velocities in the zonal and meridional directions, respectively, which are set as the control inputs before each dive. Assuming a uniform diving and surfacing speed of the glider within each dive, we aim to choose the glider heading angle, equivalently the glider's velocity components $u_{t,k}^{(m)}(x, y)$ and $v_{t,k}^{(m)}(x, y)$ with $\{(u_{t,k}^{(m)}(x, y))^2 + (v_{t,k}^{(m)}(x, y))^2\}^{1/2}$ fixed at 0.3m/s, such that the direction of the depth-integrated total velocity matches a designed glider heading direction

$$\mathbf{a}_{t,k} = \boldsymbol{\omega}_{t,k}^{(g)}(x, y) / \|\boldsymbol{\omega}_{t,k}^{(g)}(x, y)\|_2, \quad (2.2)$$

where $\boldsymbol{\omega}_{t,k}^{(g)}(x, y) = (\int_{-h}^0 u_{t,k}^{(g)}(x, y, z) dz, \int_{-h}^0 -v_{t,k}^{(g)}(x, y, z) dz)$.

To control the glider such that (2.2) is satisfied, it is critical to estimate the eddy velocity field. Note that the vertical velocity of an eddy is much weaker than its horizontal velocity (Klein et al., 2009) with maximal vertical velocity at the order of 0.05 cm/s, which is less than 1% of the horizontal velocity of the eddy and the speed of the glider. Thus, we focus on reconstructing the horizontal velocities at various depths.

2.1 Geostrophic model

To estimate the velocity field that satisfies geostrophic balance, we impose the geophysical constraint that horizontal velocities are balanced by the Coriolis force and the pressure gradient force (Vallis, 2017). When the Rossby number is small, as is typical for mesoscale eddies in the mid-latitudes, the inertial and diffusion terms in the Navier-Stokes equations become negligible, leading to the geostrophic balance equations (see Section S5 of the SM):

$$f v_t(x, y, z) - \frac{1}{\rho_0} \frac{\partial P_t(x, y, z)}{\partial x} = 0 \quad \text{and} \quad -f u_t(x, y, z) - \frac{1}{\rho_0} \frac{\partial P_t(x, y, z)}{\partial y} = 0 \quad (2.3)$$

where $P_t(x, y, z)$ is the seawater pressure, $\rho_0 = 1025 \text{ kg/m}^3$ is the reference density and f is the known Coriolis parameter. Note that the geostrophic balance expressed in (2.3) holds not only in the Eulerian coordinate system, but also in the eddy-centered coordinate system, obtained by shifting to the eddy center as (2.1). For convenience, the subsequent discussion will be conducted in the eddy-centered coordinate system.

The geostrophic balance equation in (2.3) motivates using the stream function $\Psi_t(x, y, z) := P_t(x, y, z)/(\rho_0 f)$ to express the geostrophic velocity as

$$\boldsymbol{\omega}_t(x, y, z) = (v_t(x, y, z), -u_t(x, y, z))^T = \nabla_{xy} \Psi_t(x, y, z),$$

where $\nabla_{xy} = (\partial/\partial x, \partial/\partial y)^T$ is the horizontal gradient operator. The stream function Ψ_t encodes the geostrophic velocity field and serves as the central variable in quasi-geostrophic theory (Hoskins et al., 1985). Let $\text{SSTA}_t(x, y)$ and $\text{SLA}_t(x, y)$ denote the sea surface temperature anomaly (SSTA) and the sea level anomaly. From Wang et al. (2013), the stream function at sea surface ($z = 0$) is related to SSTA and SLA through the equations

$$\Psi_t(x, y, 0) = \frac{g}{f} \text{SLA}_t(x, y) \quad \text{and} \quad \left. \frac{\partial \Psi_t(x, y, z)}{\partial z} \right|_{z=0} = \frac{b_t(x, y)}{f}, \quad (2.4)$$

where $b_t(x, y)$ is the sea surface buoyancy anomaly, approximated by $b_t(x, y) \approx (g\alpha)\text{SSTA}_t(x, y)$ with $g = 9.81$ being the gravitational acceleration constant and α being a known thermal

expansion coefficient. Meanwhile, at the sea bottom ($z = -H$), $\Psi_t(x, y, z)$ satisfies

$$\left. \frac{\partial \Psi_t(x, y, z)}{\partial z} \right|_{z=-H} = 0. \quad (2.5)$$

Based on the geostrophic balance (2.3) and the boundary conditions in (2.4) and (2.5), we develop a statistical method using a convolutional neural network (CNN) to estimate the three-dimensional current field at a given time t . The estimated velocity is then used to select the glider’s heading angle and adapt its path in a given direction via (2.2), with the details provided in Section S8 of the SM.

2.2 Multi-source data

Our study uses both historic reanalysis data derived from ocean dynamical simulations and observations, and fresh multi-type observations from satellite and in situ drifting buoys.

Reanalysis data. We use the GLORYS reanalysis data (Lellouche et al., 2018) to learn the underlying physical relationships between surface and interior velocity fields based on the geostrophic balance model, which are then applied to impute real-time interior velocity from an estimated surface velocity field. Reanalysis data are made by assimilating observations into physics models via either the ensemble Kalman filter (EnKF) or four-dimensional variational (4D-Var) methods, and are major high-quality data sources for studying ocean dynamics. GLORYS provides daily means of the key ocean state variables, including temperature, salinity, velocity, and sea level anomalies, in a spatial grid of $1/12^\circ$ horizontal resolution and 50 unevenly spaced vertical layers, which results in a high-dimensional set of $37 \times 37 \times 50 = 68450$ grids in the study region. The dataset spans from 1993 until one week before the glider experiments. GLORYS uses incremental analysis updates to introduce assimilation increments gradually as the ocean model evolves, reducing non-geophysical imbalance and retaining large-scale physical structure. Its velocity fields can also contain

departures from quasi-geostrophic balance.

Let $u_t^{(\text{glo})}(x_i, y_j, z_k)$ and $v_t^{(\text{glo})}(x_i, y_j, z_k)$ denote the zonal and meridional velocities from GLORYS at depth z_k , time t , and grid point (x_i, y_j) . The full grid domain is defined as $\mathbb{O}^{(\text{glo})} = \{(x_i, y_j, z_k) : i = 1, \dots, N_{\text{lon}}; j = 1, \dots, N_{\text{lat}}; k = 1, \dots, N_{\text{dep}}\} \subset \mathcal{D}$. The sea surface temperature anomaly $\text{SSTA}_t^{(\text{glo})}(x_i, y_j)$ of GLORYS is obtained by subtracting the long-term mean from its daily sea surface temperature field. We use GLORYS to learn the relationship between surface and interior currents for the target eddy. The transfer learning model applies this relationship to observed surface fields, allowing their marginal distribution to differ from that in the reanalysis.

Satellite–AVISO. AVISO (Archiving, Validation, and Interpretation of Satellite Oceanographic) dataset (Pascual et al., 2006) integrates observations from multiple satellite altimetry missions. It provides near-real-time (one day delay), daily-averaged SLA and global surface current fields derived from SLA based on (2.4) at a spatial resolution of $1/4^\circ$. To match the resolution of GLORYS, we interpolate AVISO data into the $1/12^\circ$ horizontal grid of GLORYS. Let $\mathbb{O}^{(\text{sat})} = \{(x_i, y_j, 0) : i = 1, \dots, N_{\text{lon}}, j = 1, \dots, N_{\text{lat}}\}$ denote the spatial surface grid of AVISO observations.

Satellite–OISST. OISST (Optimum Interpolation Sea Surface Temperature) data (Huang et al., 2021), which are widely used in oceanographic research, integrate sea surface temperature (SST) observations from multiple platforms, including satellites, ships, buoys and Argo floats, into a globally gridded SST product with the same spatial ($1/4^\circ$) and temporal (daily) resolution as the AVISO data. To match the resolution of the GLORYS dataset, we interpolate the OISST data into the same spatial grid $\mathbb{O}^{(\text{sat})}$.

In situ Buoy. While satellite altimetry can offer an estimate of the surface current field, it often fails to fully capture the true structure of the current fields due to retrieval and

interpolation errors. In contrast, drifting buoys provide direct in situ velocity measurements (Albaladejo et al., 2012). There were 20 buoys in the study mission, which passively followed the flow and transmitted real-time positions, allowing surface velocity to be estimated from their displacement over time. The limited number of buoys restricts their spatial coverage, with daily buoy observations typically covering only about one-tenth of the study area. Let $\mathbb{O}_t^{(\text{buoy})} \subset \mathbb{O}^{(\text{sat})}$ denote the set of n_t buoy grid locations at time (day) t . The buoy measurements can be used to calibrate satellite-derived sea surface currents. Part of the drifting-buoy data is used as a validation set for comparing different methods.

Different types of data have their advantages and disadvantages for estimating the velocity field of an eddy in real time. While the buoy measurements are more accurate, they are sparse in the target area. The satellite measurements cover the whole region. However, as gridded interpolated products, they may suffer from systematic bias and interpolation error. Furthermore, the observations of ocean velocity are almost exclusively limited to the surface. There is no interior velocity data that can be used for model training.

The GLORYS reanalysis data, while offering full gridded velocity fields over the entire domain, including the ocean interior, are not real-time and typically have a delay of 7 days. Together with the deviation relative to the real-time observations, those issues limit the direct application of GLORYS data to real-time glider control. Table 1 summarizes the features of the data sources and their roles in the integrated modeling for data fusion. The overall strategy is to use surface buoy and satellite measurements to accurately estimate the real-time surface currents, which are fed to a transfer learning model trained on the GLORYS reanalysis data to learn a statistical relationship between the interior and surface velocity fields, as described in Section 4. This integrated approach allows us to accurately reconstruct the interior current field in near-real time based on more readily available

Table 1: Summary of pros, cons and roles of multi-source ocean data for velocity estimation.

Data	Type	Pros	Cons	Purpose
GLORYS	Reanalysis	Fully 3D field, maintain physical relationship between ocean state variables	Subject to bias, not real time	Estimate physical relationship between interior and surface velocities
AVISO, OISST	Satellite observation	Fully observed on the ocean surface	Subject to systematic bias, no interior observation	Surface background field for velocity reconstruction
Drifting buoys	In situ observation	High accuracy at measurement locations	Sparse and non-uniform spatial coverage, no interior observation	Correct systematic biases in background fields

surface observations.

In summary, we tackle the following three scientific questions in this paper.

- Q1. How can spatially extensive but biased satellite currents be combined with sparse, more accurate buoy observations to reconstruct the real-time surface current field?
- Q2. How can quasi-geostrophic physics guide statistical learning from GLORYS to extend this reconstruction down to the glider dive depth, when in situ interior observations of the target eddy are unavailable for training?
- Q3. Can the reconstructed currents support real-time control of a glider network, and how accurately can they predict surfacing locations during the survey?

3 HD linear mixed model for surface currents

To estimate surface current, we develop a data fusion method based on a high-dimensional (HD) linear mixed model, which integrates satellite-derived velocities that are subject to

measurement bias and in situ drifting buoy observations that are only sparsely observed. The proposed method uses kernel-smoothed best linear unbiased prediction (BLUP) for the surface velocity field. In the following analysis, all velocity measurements are centered with respect to the daily eddy center as defined in (2.1).

3.1 Model for surface current and observations

Let $p = N_{\text{lon}} \times N_{\text{lat}}$ denote the number of grid points in $\mathbb{O}^{(\text{sat})}$ for the satellite observations on the ocean surface. We arrange the true ocean geostrophic velocity field (relative to eddy center) into a high-dimensional vector such that $\boldsymbol{\omega}_t = (\boldsymbol{\omega}_t(x_1, y_1, 0)^T, \dots, \boldsymbol{\omega}_t(x_{N_{\text{lon}}}, y_{N_{\text{lat}}}, 0)^T)^T \in \mathbb{R}^{2p}$. The goal is to obtain the best linear prediction of $\boldsymbol{\omega}_t$ that corrects the bias in satellite data by the sparse buoy observations. For each time t , let $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$ denote the mean and covariance matrix of $\boldsymbol{\omega}_t$, both of which would vary slowly over time due to the stability of the eddy structure. Note that the dimension of $\boldsymbol{\omega}_t$ is much larger than the number of days over the study period, making the estimation of $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$ a high-dimensional problem.

Let $\boldsymbol{\omega}_t^{(\text{sat})}$ denote the vectorized satellite-derived surface velocity in the same way as $\boldsymbol{\omega}_t$. The high-frequency velocity records from the drifting buoys are first mapped to the nearest grid points in $\mathbb{O}^{(\text{sat})}$ and then temporally averaged to match the daily resolution of the satellite data; see Section S1.2 in the SM for details. Let n_t denote the number of buoys at time t and $\boldsymbol{\omega}_t^{(\text{buoy})}$ denote the vectorized buoy velocity after recentering relative to the eddy center, where the dimension of $\boldsymbol{\omega}_t^{(\text{buoy})}$ is $2n_t$. Note that $\boldsymbol{\omega}_t^{(\text{sat})}$ has full observations on $\mathbb{O}^{(\text{sat})}$, where the dimension of $\boldsymbol{\omega}_t^{(\text{sat})}$ and $\boldsymbol{\omega}_t$ are the same. However, buoy observations are only partially observed on the grid, where $\mathbb{O}_t^{(\text{buoy})} \subset \mathbb{O}^{(\text{sat})}$ and $n_t < p$. Let $\mathbf{H}_t$ be a $2n_t \times 2p$ -dimensional observation matrix of buoys with binary values, where each row of $\mathbf{H}_t$ contains a single 1, in the column corresponding to the buoy's grid location in $\boldsymbol{\omega}_t$.

Empirical comparisons between AVISO Level-4 and in situ buoy data confirm spatially varying biases in satellite retrievals. Accordingly, to account for the biases of the gridded AVISO Level-4 data, we propose a spatially high-dimensional linear mixed effects model (HD-LMM)

$$\boldsymbol{\omega}_t^{(\text{sat})} = \boldsymbol{\omega}_t + \mathbf{G}_t \boldsymbol{\beta}_t + \mathbf{e}_t^{(\text{sat})} \quad \text{and} \quad (3.1)$$

$$\boldsymbol{\omega}_t^{(\text{buoy})} = \mathbf{H}_t \boldsymbol{\omega}_t + \mathbf{e}_t^{(\text{buoy})}, \quad (3.2)$$

where $\mathbf{b}_t^{(\text{sat})} = \mathbf{G}_t \boldsymbol{\beta}_t$ models the systematic bias of the satellite observations as a linear function of physical and environmental covariates, including longitude, latitude, and surface wind, $\mathbf{G}_t$ is the design matrix of the covariates, $\boldsymbol{\beta}_t = (\boldsymbol{\beta}_{u,t}^\top, \boldsymbol{\beta}_{v,t}^\top)^\top$ denotes the coefficient vector, and $\mathbf{e}_t^{(\text{sat})}$ and $\mathbf{e}_t^{(\text{buoy})}$ denote random measurement errors of satellite and buoy observations with zero-mean and covariance matrices $\mathbf{R}_t^{(\text{sat})}$ and $\mathbf{R}_t^{(\text{buoy})}$, respectively. We assume $\boldsymbol{\omega}_t$, $\mathbf{e}_t^{(\text{sat})}$ and $\mathbf{e}_t^{(\text{buoy})}$ to be mutually independent. Here, $\boldsymbol{\beta}_{u,t}, \boldsymbol{\beta}_{v,t} \in \mathbb{R}^m$ quantify the influence of the covariates on the zonal and meridional satellite biases at time t , respectively. The detailed formulation of the design matrix $\mathbf{G}_t$ is given in Section S6.1 of the SM.

Let $\boldsymbol{\omega}_t^{(\text{obs})} = (\boldsymbol{\omega}_t^{(\text{sat})\top}, \boldsymbol{\omega}_t^{(\text{buoy})\top})^\top$ and $\mathbf{e}_t^{(\text{obs})} = (\mathbf{e}_t^{(\text{sat})\top}, \mathbf{e}_t^{(\text{buoy})\top})^\top$. From (3.1) and (3.2),

$$\boldsymbol{\omega}_t^{(\text{obs})} = \tilde{\mathbf{H}}_t \boldsymbol{\omega}_t + \tilde{\mathbf{G}}_t \boldsymbol{\beta}_t + \mathbf{e}_t^{(\text{obs})}, \quad (3.3)$$

where $\tilde{\mathbf{H}}_t = (\mathbf{I}_{2p}, \mathbf{H}_t^\top)^\top$, $\tilde{\mathbf{G}}_t = (\mathbf{G}_t^\top, \mathbf{0}_{2m \times 2n_t})^\top$, $\mathbf{I}_{2p}$ denotes the identity matrix of dimension $2p$, and $\mathbf{0}_{2m \times 2n_t}$ denotes a matrix of zeros. The covariance matrix $\mathbf{R}_t = \text{Var}(\mathbf{e}_t^{(\text{obs})})$ forms a block-diagonal matrix with $\mathbf{R}_t^{(\text{sat})}$ and $\mathbf{R}_t^{(\text{buoy})}$ constituting its diagonal blocks. Both $\mathbf{R}_t^{(\text{sat})}$ and $\mathbf{R}_t^{(\text{buoy})}$ are assumed to be diagonal matrices, whose diagonal elements are either obtained from satellite products or estimated via replicate observations.

A key ingredient of the HD-LMM Model (3.3) is the high-dimensional covariance matrix $\boldsymbol{\Sigma}_t$. Let $\boldsymbol{\Sigma}_{uu,t}$ and $\boldsymbol{\Sigma}_{vv,t}$ be the covariance matrices corresponding to the zonal and meridional components of $\boldsymbol{\omega}_t$, respectively. Let $\boldsymbol{\Sigma}_{uv,t}$ be their cross-covariance. Then, $\boldsymbol{\Sigma}_{uu,t}$,

$\Sigma_{vv,t}$ and $\Sigma_{uv,t}$ form a 2×2 block partition of Σ_t . For locations (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}) , the entry of $\Sigma_{uu,t}$ is indexed as $\sigma_{uu,t}(i_1, j_1; i_2, j_2) = \text{Cov}(u_t(x_{i_1}, y_{j_1}, 0), u_t(x_{i_2}, y_{j_2}, 0))$. The entries of $\Sigma_{vv,t}$ and $\Sigma_{uv,t}$ can be similarly expressed. For data observed in a spatial grid, the covariance between two variables would decrease as the distance between the two variables increases, as commonly observed in geophysical data (Cressie and Huang, 1999; Genton and Kleiber, 2015). To regulate the decay in the off-diagonal entries of Σ_t with respect to the increase of spatial distance, we propose the multi-directional bandable covariance class

$$\mathcal{U}(M, \alpha, \epsilon) = \left\{ \Sigma : \epsilon \leq \lambda_{\min}(\Sigma) \leq \lambda_{\max}(\Sigma) \leq \epsilon^{-1}, \max_{i_2, j_2} \left\{ \sum_{i_1, j_1} (|\sigma_{uu}(i_1, j_1; i_2, j_2)| + |\sigma_{vv}(i_1, j_1; i_2, j_2)| + |\sigma_{uv}(i_1, j_1; i_2, j_2)|) \cdot \mathbb{1}(\Delta_{i_1 j_1, i_2 j_2}^2 > k^2) \right\} \leq M k^{-\alpha} \right\}, \quad (3.4)$$

for any $k > 0$, where M, α and ϵ are positive constants, $\Delta_{i_1 j_1, i_2 j_2}^2 = \{(x_{i_1} - x_{i_2})^2 + (y_{j_1} - y_{j_2})^2\} / (1/12^\circ)^2$ is measured in units of the $1/12^\circ$ grid spacing, and $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of Σ , respectively. The covariance class in (3.4) generalizes the bandable class for unidirectional data in Bickel and Levina (2008) to spatially bidirectional data. Compared to parametric spatial covariance models, such as the Matérn class, the proposed bandable covariance class accommodates heterogeneous and anisotropic dependence structures indicated by our empirical analysis. The sample covariances between velocities decay as spatial distance increases with different decaying rates across different components. See Figure S11 in the SM.

Compared to Nguyen et al. (2012), which uses an LMM with a parametric spatial covariance model to integrate two sets of satellite observations, our model considers measurement bias for satellite data, integrates sparsely observed in situ observations, and uses a nonparametric covariance model with a bandable structure.

3.2 Estimation of surface current

To estimate the proposed HD-LMM, we estimate the covariance matrix Σ_t through a three-stage procedure. First, an initial estimate of the bias coefficient $\hat{\beta}_t^{(1)}$ is obtained by solving the generalized least squares problem for satellite-buoy differences at locations in $\mathbb{O}_t^{(\text{buoy})}$:

$$\hat{\beta}_t^{(1)} = \underset{\beta}{\operatorname{argmin}} \|(\mathbf{H}_t \omega_t^{(\text{sat})} - \omega_t^{(\text{buoy})}) - \mathbf{H}_t \mathbf{G}_t \beta\|_{\mathbf{V}_t^{-1}}^2, \quad (3.5)$$

where $\mathbf{V}_t = \operatorname{Var}(\mathbf{H}_t \mathbf{e}_t^{(\text{sat})} - \mathbf{e}_t^{(\text{buoy})}) = \mathbf{H}_t \mathbf{R}_t^{(\text{sat})} \mathbf{H}_t^\top + \mathbf{R}_t^{(\text{buoy})}$, and $\|\mathbf{c}\|_{\mathbf{B}}^2 = \mathbf{c}^\top \mathbf{B} \mathbf{c}$ for a vector $\mathbf{c}$ and a matrix $\mathbf{B}$. Second, we calculate the bias-adjusted velocity $\tilde{\omega}_t^{(\text{sat})} = \omega_t^{(\text{sat})} - \mathbf{G}_t \hat{\beta}_t^{(1)}$, and obtain the kernel-smoothed estimates of μ_{t_0} and Σ_{t_0} as

$$\begin{aligned} \hat{\mu}_{t_0}^{(1)} &= \sum_{t=t_0-w}^{t_0} \frac{\mathcal{K}((t_0-t)/h_1)}{S_{t_0}} \tilde{\omega}_t^{(\text{sat})} \quad \text{and} \\ \hat{\Sigma}_{t_0}^{(1)} &= \sum_{t=t_0-w}^{t_0} \frac{\mathcal{K}((t_0-t)/h_1)}{S_{t_0}} \{(\tilde{\omega}_t^{(\text{sat})} - \hat{\mu}_{t_0}^{(1)})(\tilde{\omega}_t^{(\text{sat})} - \hat{\mu}_{t_0}^{(1)})^\top - \mathbf{R}_t^{(\text{sat})}\} \end{aligned} \quad (3.6)$$

in a window of size w preceding t_0 , where $S_{t_0} = \sum_{t=t_0-w}^{t_0} \mathcal{K}((t_0-t)/h_1)$, $\mathcal{K}(\cdot)$ is a one-sided boundary kernel (Jones, 1993), and h_1 is the smoothing bandwidth. Under some regularity and dependence assumptions stated in Section S6.3 of the SM, $\hat{\mu}_{t_0}^{(1)}$ and $\hat{\Sigma}_{t_0}^{(1)}$ are component-wise uniformly consistent estimates of μ_{t_0} and Σ_{t_0} , respectively.

Third, as Σ_{t_0} is a high-dimensional bandable covariance in the class $\mathcal{U}(M, \alpha, \epsilon)$, we regularize $\hat{\Sigma}_{t_0}^{(1)}$ by a localization estimator $\hat{\Sigma}_{t_0} = \hat{\Sigma}_{t_0}^{(1)} \circ \mathbf{B}_1$ to estimate Σ_{t_0} , where $\circ$ denotes the Hadamard (element-wise) product, $\mathbf{B}_1 = \mathbf{1}_{2 \times 2} \otimes \mathbf{B} \in \mathbb{R}^{2p \times 2p}$ is a 2×2 block matrix with each block component being a localization matrix $\mathbf{B} = (B_{i_1 j_1, i_2 j_2})$, and $\otimes$ denotes the Kronecker product. Specifically, each entry $B_{i_1 j_1, i_2 j_2}$ of $\mathbf{B}$ encodes the spatial dependence between locations (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}) such that

$$B_{i_1 j_1, i_2 j_2} = \exp \left(-\frac{\max\{0, \Delta_{i_1 j_1, i_2 j_2}^2 - L^2\}}{L^2} \right), \quad (3.7)$$

where L denotes the bandwidth parameter, in the same $1/12^\circ$ grid-spacing units as Δ , that governs the decay rate. The proposed covariance estimator is a special case of covariance

localization, which is commonly used in large-scale ocean data analysis (Frolov et al., 2016). The consistency of $\hat{\Sigma}_{t_0}$ is shown in Section S6.4 in the SM. The bandwidth is selected by a modified approach of Qiu and Chen (2015) using the adjusted velocities $\{\tilde{\omega}_t^{(\text{sat})}\}$; see Section S3 in the SM for details.

To predict ω_t given Σ_t , we consider a loss function

$$\mathcal{J}_t(\mu_t, \beta_t, \omega_t \mid \Sigma_t) = \|\omega_t^{(\text{obs})} - \tilde{\mathbf{H}}_t \omega_t - \tilde{\mathbf{G}}_t \beta_t\|_{\mathbf{R}_t}^2 + \|\omega_t - \mu_t\|_{\Sigma_t}^2, \quad (3.8)$$

for both the fixed and random coefficients. The loss function in (3.8) yields the BLUP under the LMM's mean and covariance assumptions. Under joint Gaussianity, it is also proportional, up to an additive constant, to the negative log joint density of $\omega_t^{(\text{obs})}$ and ω_t . For fixed μ_t , β_t , and Σ_t , its minimizer over ω_t is then the conditional expectation $\mathbb{E}(\omega_t \mid \omega_t^{(\text{obs})})$. As the fixed effects μ_t and β_t are assumed to slowly vary in time, for a given time t_0 , we incorporate data within w days preceding t_0 for the estimation. Specifically, given the localization covariance estimate $\hat{\Sigma}_t$, the boundary kernel $\mathcal{K}(\cdot)$ and the bandwidth h_2 , we solve the optimization problem with the kernel-smoothing weights

$$\{\hat{\mu}_{t_0}, \hat{\beta}_{t_0}, \hat{\omega}_{t_0}\} = \underset{\mu_{t_0}, \beta_{t_0}, \{\omega_t\}_{t=t_0-w}^{t_0}}{\operatorname{argmin}} \sum_{t=t_0-w}^{t_0} \mathcal{K}\left(\frac{t_0-t}{h_2}\right) \mathcal{J}_t(\mu_{t_0}, \beta_{t_0}, \omega_t \mid \hat{\Sigma}_t). \quad (3.9)$$

Here, the random effects $\{\omega_t\}_{t=t_0-w}^{t_0}$ are treated as distinct, while all $\{\mathcal{J}_t(\mu_{t_0}, \beta_{t_0}, \omega_t \mid \hat{\Sigma}_t)\}_{t=t_0-w}^{t_0}$ share the same fixed effects μ_{t_0} and β_{t_0} due to their slow temporal variation.

Define the residual vector $\mathbf{r}_t(\mu, \beta) = \omega_t^{(\text{obs})} - \tilde{\mathbf{H}}_t \mu - \tilde{\mathbf{G}}_t \beta$. By profile minimization, the solution to (3.9) for the daily random effects satisfies:

$$\hat{\omega}_t = \hat{\mu}_{t_0} + \hat{\Sigma}_t \tilde{\mathbf{H}}_t^T (\mathbf{R}_t + \tilde{\mathbf{H}}_t \hat{\Sigma}_t \tilde{\mathbf{H}}_t^T)^{-1} \mathbf{r}_t(\hat{\mu}_{t_0}, \hat{\beta}_{t_0}). \quad (3.10)$$

Consequently, to solve (3.9), the total loss function reduces to a local linear regression:

$$\{\hat{\mu}_{t_0}, \hat{\beta}_{t_0}\} = \underset{\mu, \beta}{\operatorname{argmin}} \sum_{t=t_0-w}^{t_0} \mathcal{K}\left(\frac{t_0-t}{h_2}\right) \{\mathbf{r}_t(\mu, \beta)^T (\tilde{\mathbf{H}}_t \hat{\Sigma}_t \tilde{\mathbf{H}}_t^T + \mathbf{R}_t)^{-1} \mathbf{r}_t(\mu, \beta)\}, \quad (3.11)$$

where $\tilde{\mathbf{H}}_t \hat{\Sigma}_t \tilde{\mathbf{H}}_t^T + \mathbf{R}_t$ is an estimate of covariance matrix of the residual $\mathbf{r}_t(\boldsymbol{\mu}_t, \boldsymbol{\beta}_t)$. Once $\hat{\boldsymbol{\mu}}_{t_0}$ and $\hat{\boldsymbol{\beta}}_{t_0}$ are obtained, they are plugged in (3.10) to obtain the velocity estimate $\hat{\boldsymbol{\omega}}_{t_0}$. The convergence rate of $\hat{\boldsymbol{\omega}}_{t_0}$ given consistent estimators of $\boldsymbol{\mu}_t$, $\boldsymbol{\beta}_t$, Σ_t is derived in the SM.

Our estimation approach incorporates a localization estimator for high-dimensional covariances that capture spatial dependencies in the velocity field of surface currents. From the form of $\hat{\boldsymbol{\omega}}_t$ in (3.10), the estimated covariance $\hat{\Sigma}_t$ plays a role analogous to that of the background error covariance in three-dimensional variational assimilation (3D-Var). The weighted minimization of the multi-day cost function simultaneously addresses satellite biases and partially-observed buoy data in a unified framework. This integrated procedure facilitates effective bias correction while optimizing the prediction accuracy.

4 Physics-guided learning for interior currents

In this section, we develop a physics-guided convolutional neural network model for estimating the relationship between the interior velocity field and the surface velocity field, grounded in the quasi-geostrophic theory. The GLORYS reanalysis data are used to fit the neural network, and the fitted model is applied to the estimated surface velocities obtained in Section 3 to estimate the interior velocity. This is in the same spirit as transfer learning, which assumes that the reanalysis data keep a valid physical relationship of the current velocity between the surface and interior, although its marginal distribution of surface velocities may not be accurate and subject to bias compared to in situ observations.

Recall that $\text{SSTA}_t(x, y)$ denotes the sea surface temperature anomaly under the eddy-centered coordinate system. We first introduce the physics-guided statistical model for estimating interior velocities $u_t(x, y, z)$ and $v_t(x, y, z)$ for $z < 0$ based on (2.3). As shown in Section S7 of the SM, under the quasi-geostrophic framework, the stream function can

be expressed, up to a horizontally constant component, as

$$\Psi_t(x, y, z) = \iint_{\mathcal{D}_x \times \mathcal{D}_y} G(x - x', y - y', z) \text{SSTA}_t(x', y') dx' dy' + \sum_{i=0}^{\infty} \theta_{t,i}(x, y) F_i(z), \quad (4.1)$$

where $G(x, y, z)$ is the Green's function, defined in (S7.6) in the SM, and can be viewed as a locally smoothing operator, $\{F_i(z)\}_{i=0}^{\infty}$ are known orthogonal eigenfunctions of the Sturm-Liouville equation defined in (S7.8) of the SM, and $\{\theta_{t,i}(x, y)\}_{i=0}^{\infty}$ are the corresponding coefficient functions. Differentiating both sides of (4.1), we have

$$\omega_t(x, y, z) = \iint \nabla_{xy} G(x - x', y - y', z) \text{SSTA}_t(x', y') dx' dy' + \sum_{i=0}^{\infty} \nabla_{xy} \theta_{t,i}(x, y) F_i(z). \quad (4.2)$$

Hence, the surface geostrophic current $\omega_t(x, y, 0)$ satisfies

$$\omega_t(x, y, 0) = \iint \nabla_{xy} G(x - x', y - y', 0) \text{SSTA}_t(x', y') dx' dy' + \sum_{i=0}^{\infty} \nabla_{xy} \theta_{t,i}(x, y) F_i(0). \quad (4.3)$$

From Equations (4.2) and (4.3), the surface currents $\omega_t(x, y, 0)$ and interior currents $\omega_t(x, y, z)$ share the same coefficients $\{\theta_{t,i}(x, y)\}$ for the solution of the geostrophic model. This leads to the following statistical model for $\omega_t(x, y, z)$ with respect to $\omega_t(x, y, 0)$ and $\text{SSTA}_t(x, y)$:

$$\omega_t(x, y, z) = \Phi_t(\omega_t(\cdot, \cdot, 0), \text{SSTA}_t(\cdot, \cdot))(x, y, z) + \xi_t(x, y, z), \quad (4.4)$$

which links the eddy surface currents and temperatures to the eddy interior velocity fields, where we denote $\omega_t(\cdot, \cdot, 0) = \{\omega_t(x', y', 0) : (x', y') \in \mathbb{O}^{(\text{sat})}\}$ and $\text{SSTA}_t(\cdot, \cdot) = \{\text{SSTA}_t(x', y') : (x', y') \in \mathbb{O}^{(\text{sat})}\}$ for simplicity. Here, $\Phi_t(\omega_t(\cdot, \cdot, 0), \text{SSTA}_t(\cdot, \cdot))(x, y, z)$ is the conditional mean of $\omega_t(x, y, z)$ given $\omega_t(\cdot, \cdot, 0)$ and $\text{SSTA}_t(\cdot, \cdot)$, which is slowly varying over time. The learned relationship of (4.4) from the reanalysis data will allow estimating the interior currents using the estimated surface current and temperature fields.

As ocean dynamics change slowly over time, it is reasonable to assume a slowly time-varying model Φ_t in (4.4). For example, time-invariant relationships are assumed in the

subsurface thermal anomalies modeling (Zhang et al., 2024) and data-driven weather forecasting (Bi et al., 2023). In our setting, this assumption is justified as long as the coefficients $\{\theta_{t,i}\}$ are slowly changing over time t , which naturally aligns with the potential vorticity conservation property (Early et al., 2011) of mesoscale eddies that preserve their internal dynamical structures over a period of time.

The following proposition shows that under a spatial Gaussian field prior on the coefficients $\{\theta_{t,i}(x, y)\}$, the regression function $\Phi_t(\cdot, \cdot)$ is linear.

Proposition 1 *If $\{\theta_{t,i}(x, y)\}_{i=0}^\infty$ are independent zero-mean Gaussian processes with twice differentiable covariance functions $\{\mathcal{C}_i\}_{i=0}^\infty$, are independent of $\text{SSTA}_t(\cdot, \cdot)$, and the velocity modal series in (4.2) converge in mean square, then (4.4) holds with $\xi_t(x, y, z)$ being a zero-mean Gaussian process, independent of the surface fields $\omega_t(\cdot, \cdot, 0)$ and $\text{SSTA}_t(\cdot, \cdot)$, and Φ_t is linear in that*

$$\begin{aligned} \Phi_t(\omega_t(\cdot, \cdot, 0), \text{SSTA}_t(\cdot, \cdot))(x, y, z) &= \int_{\mathcal{D}_x \times \mathcal{D}_y} \mathcal{T}_z(x, y; x', y') \omega_t(x', y', 0) dx' dy' \\ &+ \int_{\mathcal{D}_x \times \mathcal{D}_y} \{\nabla_{xy} G(x - x', y - y', z) - \tilde{\mathcal{T}}_z(x, y; x', y')\} \text{SSTA}_t(x', y') dx' dy', \end{aligned} \quad (4.5)$$

where the regression coefficient functions $\mathcal{T}_z$ and $\tilde{\mathcal{T}}_z$ are determined by the prior covariance function $\{\mathcal{C}_i\}$, and their specific forms are derived in Section S7 in the SM.

Proposition 1 derives a form for the conditional mean of $\omega_t(x, y, z)$ given $\omega_t(\cdot, \cdot, 0)$ and $\text{SSTA}_t(\cdot, \cdot)$ under a Gaussian field prior on $\{\theta_{t,i}(x, y)\}$. If the covariance functions $\{\mathcal{C}_i\}$ are spatially bandable, the coefficient functions $\mathcal{T}_z$ and $\tilde{\mathcal{T}}_z$ serve as locally smoothing weights, while the Green's function kernel $G(x, y, z)$ induces localized influence regions in the predictor space. This reveals an inherent local regression structure. Pointwise regression has been used to learn subsurface thermal anomalies from surface variables (Zhang et al., 2024). For velocity projections, the corresponding regression conditional on SSTA yields

an explicit expression for $R^2(z)$ (Lemma S4 in Section S7.3 of the SM). Proposition 1 motivates a field-to-field regression that also incorporates the surrounding surface currents and temperatures.

To model the local structure, we adopt a convolutional neural network (CNN, Krizhevsky et al. 2012) to fit $\Phi_t^{(\text{cnn})}(\cdot, \cdot; \vartheta)$ as a physics-inspired nonparametric extension of (4.5) to represent the regression function $\Phi_t(\cdot, \cdot)$ in (4.4), where ϑ denotes the parameters of the CNN. CNN mimics the localized action of $G(x, y, z)$ and the smoothing weights, but with the flexibility to capture more complex nonlinear dependencies when Gaussian priors or linear assumptions are inadequate. The CNN architecture consists of four convolutional blocks: the first three use 3×3 kernels with 32, 64, and 128 channels, respectively. Each block is followed by batch normalization, ReLU activation, and a 10% dropout layer. A final 1×1 convolution maps the feature representation to $2(N_{\text{dep}} - 1) = 98$ output channels, corresponding to the interior layers of $u_t(x, y, z)$ and $v_t(x, y, z)$. Compared to the point-wise or functional linear models, CNNs can learn rich hierarchical spatial features, making them well-suited to geophysical problems governed by localized physical interactions.

The transfer learning model assumes that the conditional mean relationship between surface fields and interior currents is shared by GLORYS and the target eddy, while their marginal distributions of surface fields may differ. This is the covariate-shift assumption. We train the model on GLORYS and apply it to the estimated surface currents and observed surface temperatures.

For a given date t_0 , considering the 7-day delay of the GLORYS analysis product, we select the 55-day half-open window $[t_0 - 62, t_0 - 7)$ for training. The 6-hourly snapshots in this window yield 220 training samples at $1/12^\circ$ spatial resolution and spatially aligned to the eddy center. The model is trained on the 6-hourly samples, and early stopping is

determined using the corresponding daily GLORYS analysis data as the validation set to prevent overfitting. The CNN $\Phi_t^{(\text{cnn})}(\cdot, \cdot; \vartheta)$ is fitted by minimizing a locally weighted mean squared error (MSE) loss:

$$\mathcal{L}_{t_0}(\vartheta) = \sum_{t \in \mathcal{N}(t_0)} \mathcal{K}\left(\frac{t_0 - t}{h_3}\right) \sum_{(x,y) \in \mathbb{O}^{(\text{sat})}} \sum_{k=2}^{N_{\text{dep}}} \left\{ \left\| \Phi_{t_0}^{(\text{cnn})}(\omega_t^{(\text{glo})}(\cdot, \cdot, 0), \text{SSTA}_t^{(\text{glo})}(\cdot, \cdot); \vartheta)(x, y, z_k) - \omega_t^{(\text{glo})}(x, y, z_k) \right\|^2 \right\}, \quad (4.6)$$

where $\mathcal{N}(t_0) = \{t_0 - 62 + k/4 : k = 0, \dots, 219\}$, $\mathcal{K}$ is the same one-sided boundary kernel used in Section 3, and h_3 is the bandwidth parameter for the CNN loss function. Training is conducted on an NVIDIA RTX 4090 GPU for up to 5000 epochs using the Adam optimizer (learning rate 10^{-3} , weight decay 10^{-5}), with ϑ updated via gradient descent to minimize $\mathcal{L}_{t_0}(\vartheta)$. The fitted model achieving the lowest validation loss is then applied to the estimated ocean surface velocities introduced in Section 3.2.

Given the fitted model $\hat{\Phi}_t^{(\text{cnn})}(\cdot, \cdot; \hat{\vartheta})$, the proposed estimate of the interior velocity field over the domain of the GLORYS reanalysis data $\mathbb{O}^{(\text{glo})}$ at day t is

$$\{\hat{\Phi}_t^{(\text{cnn})}(\hat{\omega}_t(\cdot, \cdot, 0), \text{SSTA}_t^{(\text{oisst})}(\cdot, \cdot); \hat{\vartheta})(x, y, z) : (x, y, z) \in \mathbb{O}^{(\text{glo})}\}, \quad (4.7)$$

where $\hat{\omega}_t = \hat{\mu}_t + \hat{\Sigma}_t \tilde{\mathbf{H}}_t^T (\mathbf{R}_t + \tilde{\mathbf{H}}_t \hat{\Sigma}_t \tilde{\mathbf{H}}_t^T)^{-1} \mathbf{r}_t(\hat{\mu}_t, \hat{\beta}_t)$ is the estimated surface velocities from (3.10) and (3.11), and $\text{SSTA}_t^{(\text{oisst})}$ is the sea surface temperature anomaly from the OISST satellite data. The proposed method can capture the physical relationships between sub-surface ocean currents and their surface counterparts as well as sea surface temperature, while allowing for slow temporal variations in these relationships.

5 Results

This section evaluates the reconstructed velocity fields and reports glider control results based on the estimated currents during our eddy observation campaign. The evaluation

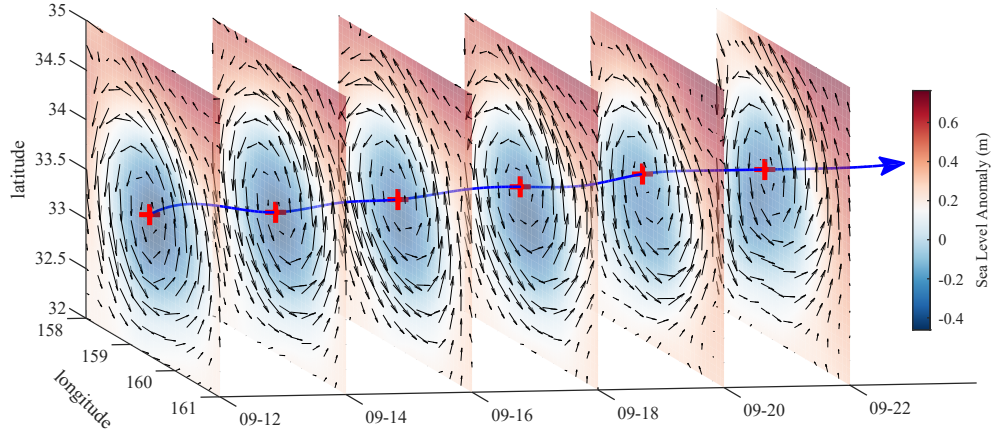


Figure 2: Sea level anomaly (represented by color) and temporal evolution of the reconstructed eddy surface current by the proposed HD-LMM method (represented by arrow). The red plus signs mark the eddy center.

of the proposed current estimation method was conducted based on in situ drifting-buoy data not used in training and independent shipborne Acoustic Doppler Current Profiler (ADCP, a device used to measure water currents) observations. ADCP provides an independent reference for interior velocity evaluation along the ship route. Using the proposed data fusion methods, we achieved the first successful control of seven underwater gliders, obtaining valuable hydrographic data on the eddy.

Figure 2 shows the temporal evolution of the reconstructed surface current field, obtained by combining satellite altimetry and buoy observations after bias correction via the HD-LMM. The average relative bias between the current fields of the buoys and the AVISO products is 44.6% after removing weak AVISO currents less than 0.3m/s in order to make the relative bias stable. The regression of $\mathbf{H}_t \boldsymbol{\omega}_t^{(\text{sat})} - \boldsymbol{\omega}_t^{(\text{buoy})}$, the velocity difference between AVISO and buoys, on $\mathbf{H}_t \mathbf{G}_t$ via (3.5) has R^2 of 80% on average, which shows that the HD-LMM captures much of the spatially varying bias of the AVISO product.

5.1 Validation on drifting buoy and ADCP

We evaluated the quality of the reconstructed ocean current fields using validation data from drifter buoys and ship-carried ADCP equipment. We evaluated the reconstructed surface velocity using five-fold cross-validation of the drifting-buoy observations. Specifically, on each day, each fold was used in turn to evaluate the accuracy of the estimated surface current field using the other four folds.

We compared the estimated surface current with the GLORYS reanalysis field and AVISO satellite surface current estimates. As a buoy-only baseline, we also obtained the Kriging estimates solely based on the training buoy data using the Matérn-class covariance with estimated parameters. Table 2 summarizes the five-fold Cross-validation Root Mean Square Error (RMSE) (rmse_u and rmse_v) of the estimated zonal and meridional velocities $\hat{u}_t(x, y, 0)$ and $\hat{v}_t(x, y, 0)$, respectively, and the total velocity errors. For each buoy validation set,

$$\text{rmse}_u = \left(\frac{\sum_{t \in \mathcal{T}^{(\text{buoy})}} \sum_{(x,y) \in \mathbb{O}_t^{(\text{buoy})}} \{u_t^{(\text{buoy})}(x, y, 0) - \hat{u}_t(x, y, 0)\}^2}{\sum_{t \in \mathcal{T}^{(\text{buoy})}} |\mathbb{O}_t^{(\text{buoy})}|} \right)^{1/2}, \quad (5.1)$$

where $\mathcal{T}^{(\text{buoy})}$ denotes the set of days and $\mathbb{O}_t^{(\text{buoy})}$ denotes set of grid locations of the validation buoys on day t , and that for the meridional velocity error rmse_v is defined analogously. The total velocity error is computed as $\text{rmse}_{uv} = (\text{rmse}_u^2 + \text{rmse}_v^2)^{1/2}$. Daily RMSE measures can be constructed by restricting $\mathcal{T}^{(\text{buoy})}$ to a date t in (5.1).

Table 2 shows that the proposed data-fusion based surface velocity field has the lowest validation RMSEs in all three measures, with the total velocity error rmse_{uv} being only 22% of the GLORYS, 32% of the AVISO, and 63% of the Kriging. These indicate the advantage of the proposed data fusion procedure in attaining more accurate estimation of surface current. The larger GLORYS errors may reflect limited observational information about the real-time state of the target eddy. AVISO directly incorporates satellite observations

of the ocean surface, with spatially varying biases addressed by the HD-LMM.

Table 2: Root mean square errors of the surface zonal (rmse_u) and meridional (rmse_v) velocity fields based on five-fold cross-validation of surface drifting-buoy observations from September 10 to 24, 2024. Values are reported as $\text{rmse} \pm \text{SD}$ (standard deviation). The total velocity error is $\text{rmse}_{uv} = (\text{rmse}_u^2 + \text{rmse}_v^2)^{1/2}$.

Method	$\text{rmse}_u \pm \text{SD}_u$ (m/s)	$\text{rmse}_v \pm \text{SD}_v$ (m/s)	$\text{rmse}_{uv} \pm \text{SD}_{uv}$ (m/s)
GLORYS	0.2056 ± 0.0868	0.2716 ± 0.0596	0.3406 ± 0.0607
AVISO	0.1507 ± 0.0455	0.1773 ± 0.0488	0.2327 ± 0.0257
Kriging on buoy	0.0796 ± 0.0174	0.0877 ± 0.0256	0.1184 ± 0.0217
Proposed (HD-LMM)	0.0494 ± 0.0110	0.0561 ± 0.0167	0.0747 ± 0.0150

Figure 3(a) presents the daily total RMSEs $\text{rmse}_{uv,t}$ of estimated velocities by the same five-fold cross-validation, with shaded areas representing the error bands of the RMSEs, computed as $(\text{mse}_{uv,t} \pm 2 \times \text{se}_{uv,t})^{1/2}$, where $\text{mse}_{uv,t} = \text{rmse}_{uv,t}^2$ and $\text{se}_{uv,t}$ denote the standard error of $\{u_t^{(\text{buoy})}(x, y, 0) - \hat{u}_t(x, y, 0)\}^2 + \{v_t^{(\text{buoy})}(x, y, 0) - \hat{v}_t(x, y, 0)\}^2$ across testing buoy observations on day t . The proposed current estimates consistently outperform the GLORYS, the AVISO, and the buoy-based kriging estimates. Among these benchmarks, GLORYS has the largest errors, followed by AVISO and buoy-based kriging. The lower errors relative to both satellite-only and buoy-only estimates show the benefit of combining broad satellite coverage with accurate in situ measurements for real-time surface-current reconstruction (Q1).

We next evaluate the proposed interior velocity field estimates in (4.7) using the ADCP measurements. We compare the reconstructed three-dimensional current field with those from the GLORYS, the climatological state, and the physics-based reconstruction method

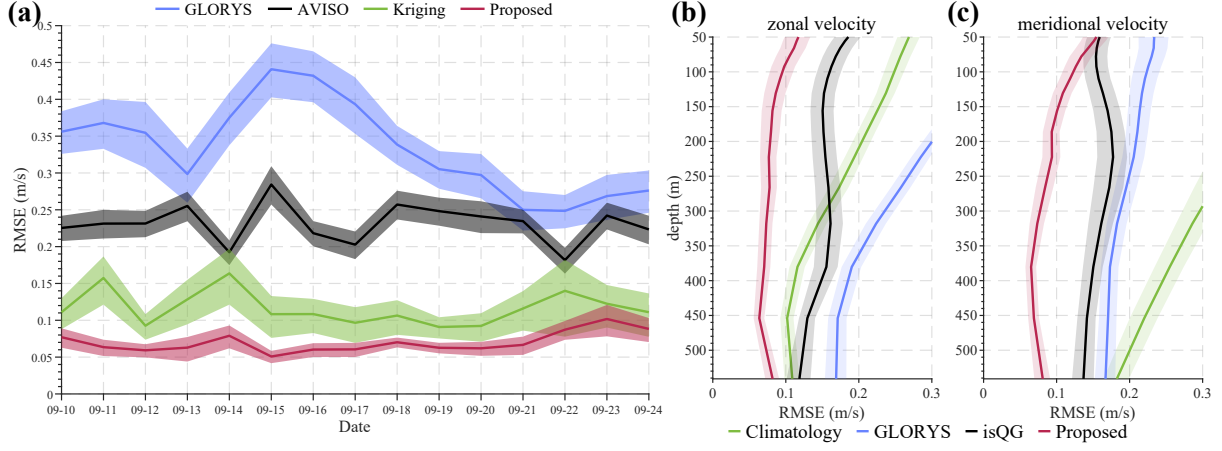


Figure 3: Daily RMSEs of estimated surface velocities by the five-fold cross-validation of the drifting buoys data in Panel (a) and the depthwise testing RMSEs in the zonal (Panel (b)) and meridional (Panel (c)) components by the ADCP observations.

isQG (Wang et al., 2013). During the field campaign, ADCP instruments were deployed to obtain velocity profiles along the ship’s route on September 12, 14, and 15. ADCP calculates the currents by sending sound pulses into the water and using the Doppler effect to detect the frequency shifts of the reflected sound waves (Herbers and Lentz, 2010). The locations and preprocessing procedure of the ADCP data are detailed in Figure S14 and Section S1.3 of the SM, respectively. The overall vertical error profiles in the zonal and meridional directions, calculated in the same way as (5.1), are presented in Figure 3(b)-(c), while the evaluation on each day is reported in Figures S15–S17 in the SM. From these figures, it can be seen that the proposed reconstruction outperforms the other methods across all depths. Notably, GLORYS performs worse than the climatology velocity in the zonal component, while the isQG method shows improvement over the GLORYS and climatology velocity, but remains inferior to the proposed approach. The independent ADCP comparisons support the accuracy of the physics-guided reconstruction at the sampled locations and depths. This agreement is obtained using surface inputs and a relationship learned from GLORYS,

without using the target eddy’s in situ interior observations for training (Q2).

5.2 Realized glider path and prediction error

In this subsection, we report the realized paths of seven gliders during the operation, which were controlled by the approach in (2.2) using manually given directions $\{\mathbf{a}_{t,k}\}$ and the proposed estimates of the three-dimensional velocity field. The criterion for selecting the glider target headings $\{\mathbf{a}_{t,k}\}$ is to maximize the coverage of the eddy, given constraints on the releasing and final retrieval locations of gliders. The rationale was to make each glider travel in the tangent direction of ocean current for a speedy surveying of the eddy at minimum energy usage. The designed pathways are reported in Figure S13 in the SM.

The realized trajectories are shown in Figure 4(a). The operation included the late release of Glider 2, misdirection of Glider 5, and early retrieval of Gliders 2 and 6. The gliders largely traveled along the designated paths and covered the eddy as shown (Figure 4(b)). Among the seven gliders, the longest operational duration was made by Glider 1, which operated at the edge of the eddy for a total of 179 hours and collected 47 vertical profile observations in 404.6 km surface trajectory. In contrast, Glider 6 had the shortest duration, operating for 130 hours and collecting 31 profiles, due to being deployed late and recovered earlier. Interestingly, although Glider 5 attained 38 profiles, as it traveled near the eddy center under weak currents, it traveled the shortest surface distance of 214.49 km. See Table S3 in the SM for more information.

The eddy occupied a region of 180 km in longitude and 167 km in latitude. Figure 4(b) displays the surface sampling gaps (SG) of the gliders, defined as the Euclidean distance to the nearest gliders’ trajectory for any location within the eddy surface. The trajectories of the seven gliders produced 261 profiles (dives) in total, yielding a maximum sampling

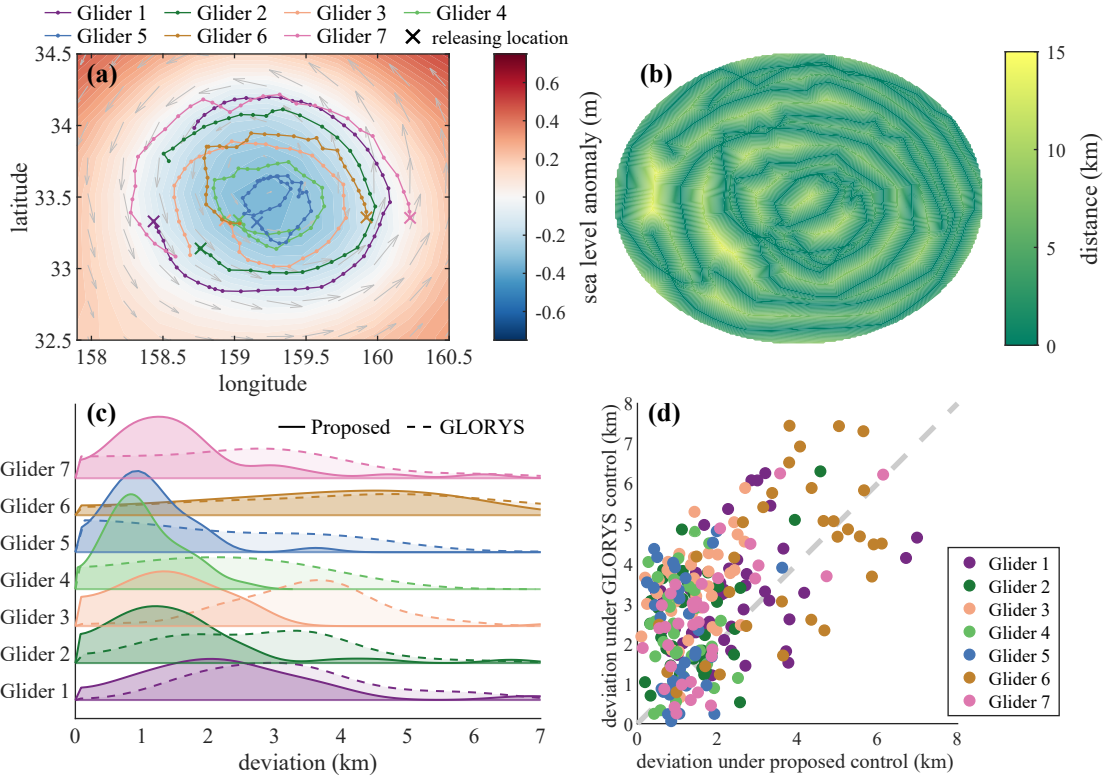


Figure 4: (a) Realized glider pathways by the proposed method, (b) heatmap of sampling gaps (distance to the closest trajectory) of the realized pathways. Panels (c) and (d) are the kernel densities and scatter plots of the deviations between the predicted and actual surfacing points based on the proposed and the GLORYS reanalysis current fields.

gap (MSG) of 14.12 km and an average sampling gap (ASG) of 3.53 km. The designed trajectories shown in Figure S13 used 288 profiles to achieve an ASG of 3.13 km and an MSG of 12.72 km. The realized pathways had slightly larger sampling gaps with 261 profiles under varying ocean currents. Early recovery of some gliders limited coverage of their planned paths. Collectively, the mission yielded the first fine-scale, high-resolution, three-dimensional temperature and salinity observations within an ocean eddy, providing valuable data for oceanographic research.

We evaluated navigation error as the distance between the predicted and actual surfacing locations for each glider profile, using current fields from the proposed method and

GLORYS reanalysis; see Section S8 of the SM for details. Figure 4(c) presents the kernel density estimates of the surfacing point deviation of the seven gliders. With the exception of Glider 6, the proposed method consistently produces much lower deviation than the GLORYS. Figure 4(d) further displays the overall surfacing point deviations using the proposed and GLORYS current estimates, indicating that the proposed method achieves smaller errors in more than three-quarters of the cases. The prediction error under GLORYS is approximately 2.47 times that under our proposed method. Together, the realized survey coverage and the smaller surfacing-location prediction errors support the use of the reconstructed currents for real-time glider control (Q3). To quantify the uncertainty associated with the one-step prediction of the glider surfacing location, we drew an independent ensemble $\{\hat{\omega}_{t,\ell}\}$ from the estimated conditional distribution of the surface velocity ω_t given the observed data $\omega_t^{(\text{obs})}$. For each ℓ , we applied the fitted CNN model to the ensemble $\{\hat{\omega}_{t,\ell}\}$ according to (4.7) to obtain an ensemble of estimated interior velocity fields at day t . These ensembles reflect the uncertainty in velocity estimation and were used to construct a prediction region for the next surfacing location of a glider; see Section S9 of the SM for details.

5.3 Sensitivity analysis and ablation study

The proposed ocean current reconstruction method is based on a set of physical and statistical assumptions, including the geostrophic approximation in (2.3), the removal of high-frequency variability in buoy velocities by daily averaging, the localized covariance structure in (3.4) and the stability of physical constraints in the CNN training in (4.6). To assess the robustness to these assumptions, we conducted sensitivity analyses and ablation studies.

We first assessed the sensitivity of the surface reconstruction to ageostrophic motions

in the drifting-buoy data. After applying an additional low-pass filter to the buoy velocities before constructing the daily gridded buoy product, the reconstructed daily surface current field differs by only $1.39\% \pm 0.22\%$ (mean $\pm$ standard deviation); see Section S4.1 for details. This indicates that including the high-frequency ageostrophic component has minimal effect on the reconstructed daily-averaged surface current.

We then considered two alternative variants of the subsurface reconstruction: (i) a geostrophic modification, which replaced the original full GLORYS current with the GLORYS geostrophic current, obtained via the geostrophic balance at the surface together with the thermal-wind relationship in the ocean interior, and (ii) a divergence modification, which added an explicit divergence penalty $\sum_{x,y,z} \{\nabla_H \cdot \Phi_{t_0}^{(\text{cnn})}(\cdot, \cdot; \vartheta)(x, y, z)\}^2$ to the CNN loss, where $\nabla_H \cdot \mathbf{a}(x, y, z) = \partial_x a_1(x, y, z) + \partial_y a_2(x, y, z)$ denotes the horizontal divergence for $\mathbf{a}(x, y, z) = (a_1(x, y, z), a_2(x, y, z))^T$. Details of these two variants are in Section S4.2 of the SM. As shown in Figure S6, the geostrophic modification leads to a modest reduction in divergence while maintaining comparable RMSEs against the independent ADCP validation data, thereby yielding a slightly more geostrophically balanced reconstruction without sacrificing prediction accuracy. In contrast, the divergence modification produces a much larger reduction in divergence, but at the cost of an increased RMSE and a reduced vorticity. These suggest that stronger regularization can improve physical balance, but does not necessarily improve predictive accuracy in the present setting.

We also conducted ablation studies on the surface and subsurface reconstruction. For the surface reconstruction, we examined the main ingredients of the HD-LMM by removing them one at a time. Specifically, we compared the full model with three reduced versions: one without the satellite-bias correction, one without covariance localization, and one without both. The 5-fold cross-validated RMSEs using the buoy data reported in Table S2 show

that removing either component worsens the prediction accuracy with the largest degradation occurring when the bias-correction is excluded. For the subsurface reconstruction, we varied the temporal-adaptation parameter h_3 in (4.6). The proposed method generally outperforms the equally weighted case when h_3 is chosen in a reasonable range; see Section S4.4 for details.

Finally, as the proposed glider-control procedure is based on daily-averaged currents, we examined whether ignoring sub-daily variation could induce bias in glider navigation. Figure S18 shows that the sub-daily ageostrophic variability is relatively strong only near the surface, but its practical impact on glider navigation is limited as the gliders spend most of their time below the surface layer. Figure S19 shows the bivariate errors between the one-step predicted and realized surfacing locations under the daily-averaged velocity field. A one-sample Hotelling T^2 test does not reject the null hypothesis that the mean bivariate prediction error is zero ($p = 0.572$); see Section S4.5 for details.

6 Discussion

Our results show that the proposed physics-guided statistical data fusion method can substantially improve both surface and subsurface current reconstruction for real-time glider operations in the target eddy. How to incorporate physical knowledge and how to model high-dimensional covariance structure are two central issues in multi-source data fusion for Earth science. In the present work, physical knowledge is introduced through model construction. Covariance localization is widely used in data assimilation, especially in high-dimensional settings where the classical sample covariance is inconsistent. The multi-directional bandable covariance estimator and its theoretical justification provide new insights in this direction. How to combine physical knowledge with high-dimensional data

under more general settings is an important direction for future research.

The methods developed in this paper may be useful beyond oceanography. The observation paradigm of “dense-but-biased” and “sparse-but-accurate” is common in other problems, where broad spatial coverage must often be combined with limited but more accurate in situ observations. Examples include the evaluation and calibration of satellite observations or physics-based model outputs using in situ observations (Albergel et al., 2012; Li et al., 2017). From this perspective, the HD-LMM for bias correction can be used for the fusion of multiple heterogeneous data sources. Moreover, using physical laws to guide statistical modeling and inference can be applied to other AI-for-Science problems that integrate the physics of a dynamic system with observations to attain fusion of model and data, not just fusion among data.

Code Availability. The code and data used to reproduce the results in this paper are publicly available at <https://zenodo.org/records/19434016>.

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Conflict of interest. The authors report there are no competing interests to declare.

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